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Notes on
Rankine's Civil Engineering

after the Notes of
Prof's. William Allan & G. W. C. Lee
by

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Preface

Rankine's Civil Engineering presents many difficulties to the student, and the Notes here presented were written primarily for use in the author's classes, and secondarily with the hope that they may be equally useful to students elsewhere.

An effort is made to have the figures and equations speak for themselves thus requireing the use of but few explanatory words, and the figures are drawn with great care since it is thought that a good figure is the best aid that can be given to any one who wishes to clearly understand an engineering problem.

The late William Allan M.A. Prof. of Applied Math. at Washington & Lee University, subsequently Principal of the McDonogh School, published in 1873, "Notes on Rankine's Applied Mechanics Part I and Rankine's Civil Engineering Parts II & III".

G.W.C. Lee L.L.D., now President of Washington & Lee University, while teaching Engineering at the Virginia Military Institute, from 1865 to 1870, made copious notes on Rankine's Civil Engineering.

The work of both these men, my predecessors in teaching Civil Engineering at Washington & Lee University, is made the basis of the Notes here presented, which also contain much original matter.

Where an expression or equation is put in brackets made thus { } it means that it has been assumed, either as some well known formula from calculus or elsewhere, or as known from some previous discussion in the text or Notes; the place from which it is taken is usually indicated.

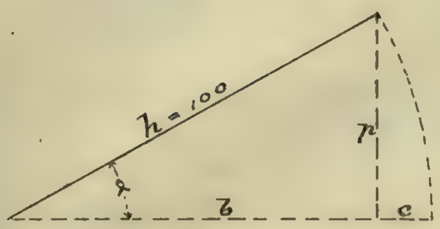
p 164 } Means that the notes immediately following are
§ 107 } about what is found on page 164, article 107 of the text.

For *Errata* both to the 19th edition 1894 of the text, and to these Notes see back of the book pp. 183-4.

Lexington, Va. Dec. 1, 1894. David C. Humphreys

Rankine's Civil Engineering.
Part 1.

p.20
§23



c = correction in links pr. chain
α = inclination
p = fall in links pr. ch.

$\therefore$ $c = 100 \text{ versin}\,\alpha$ ..... (1) ;
$c = 100 - \sqrt{10{,}000 - p^2}$ ..... (2)

$h^2 - b^2 = p^2 \quad \therefore (h-b)(h+b) = p^2 \quad \therefore h - b = \dfrac{p^2}{h+b} = c$

$\therefore c = \dfrac{p^2}{100+b} = \dfrac{p^2}{200}$ nearly 3

when α is very small.

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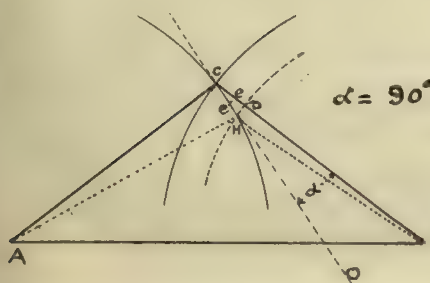
§26



$\alpha = 90° - ACB$

e = CD = error made in measuring BC
e' = CH = amt. the point c is thrown out.

then $e' = e \sec\alpha = e \operatorname{cosec} ACB$
if $ACB = 90$, $\alpha = 0$, $e' = e$
" $ACB = 30$, $\alpha = 60$, $\sec\alpha = 2$, $e' = 2e$
" $ACB = 150$, $\alpha = -60$, $\sec\alpha = -2$, $e' = -2e$

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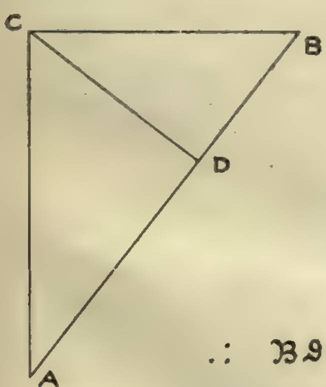
p.24
§26



$\cos BDC = -\cos ADC.$

$$\left.\begin{aligned} AC^2 &= AD^2 + CD^2 - 2\,AD\cdot CD \cos ADC \\ BC^2 &= BD^2 + CD^2 + 2\,BD\cdot CD \cos ADC \end{aligned}\right\} \therefore$$

$$BD\cdot AC^2 = BD\cdot AD^2 + BD\cdot CD^2 - 2\,AD\cdot BD\cdot CD \cos ADC$$
$$AD\cdot BC^2 = AD\cdot BD^2 + AD\,CD^2 + 2\,AD\cdot BD\cdot CD \cos ADC$$

$$\therefore BD\cdot AC^2 + AD\cdot BC^2 = BD\cdot AD^2 + AD\cdot BD^2 + (BD + AD)\,CD^2$$

$$CD^2 = \frac{AC^2\cdot BD + BC^2\cdot AD - (AD^2\cdot BD + AD\cdot BD^2)}{AD + BD}$$

$$CD^2 = \frac{AC^2\cdot BD + BC^2\cdot AD}{AB} - AD\cdot BD \text{ .... } (1)$$

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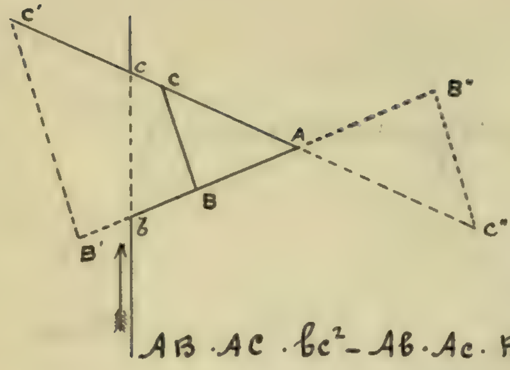
§27



$$\left.\begin{array}{l} bc^2 = Ab^2 + Ac^2 - 2Ab\cdot Ac \cos BAC \\ BC^2 = AB^2 + AC^2 - 2AB\cdot AC \cos BAC \end{array}\right\} \therefore$$

$$\left.\begin{array}{l} AB\cdot AC\cdot bc^2 = AB\cdot AC(Ab^2 + Ac^2 - \text{etc.} \\ Ab\cdot Ac\cdot BC^2 = Ab\cdot Ac(AB^2 + AC^2 - \text{etc.} \end{array}\right\} \therefore$$

$$AB\cdot AC\cdot bc^2 - Ab\cdot Ac\cdot BC^2 = AB\cdot AC(Ab^2+Ac^2) - Ab\cdot Ac(AB^2+AC^2) \quad \therefore$$

$$bc = \sqrt{Ab^2 + Ac^2 - \frac{Ab\cdot Ac}{AB\cdot AC}(AB^2 + AC^2 - BC^2)} \quad \ldots\ldots\ldots\ldots (3)$$

$$\left.\begin{array}{l} Ab = \dfrac{Ab+Ac}{2} + \dfrac{Ab-Ac}{2} \\ Ac = \dfrac{Ab+Ac}{2} - \dfrac{Ab-Ac}{2} \end{array}\right\} \therefore \left\{\begin{array}{l} Ab\cdot Ac = \dfrac{(Ab+Ac)^2 - (Ab-Ac)^2}{4} \\ AB\cdot AC = \dfrac{(AB+AC)^2 - (AB-AC)^2}{4} \end{array}\right\} \therefore$$

$$bc = \sqrt{Ab^2 + Ac^2 - \frac{(Ab+Ac)^2 - (Ab-Ac)^2}{(AB+AC)^2 - (AB-AC)^2}(AB^2 + AC^2 - BC^2)} \quad \ldots\ldots (3\,A)$$

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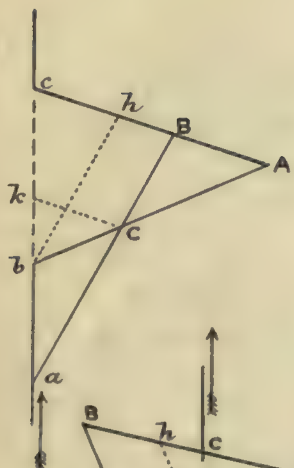
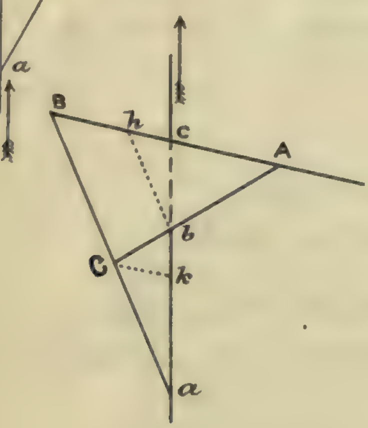
§27





$$Ck : Cb :: Ac : Ab$$
$$Ca : Ck :: Ba : Bc$$
$$\therefore \quad Ck\cdot Ca : Cb\cdot Ck :: Ac\cdot Ba : Ab\cdot Bc$$
$$Ca\cdot Ab\cdot Bc = Cb\cdot Ac\cdot Ba$$

but $Ac = AB \pm Bc$

$\therefore$

$$Bc\cdot Ca\cdot Ab = AB\cdot aB\cdot bC \pm Bc\cdot aB\cdot bC$$

$$\therefore \quad Bc = \frac{AB\cdot aB\cdot bC}{Bc\cdot Ca \mp aB\cdot bC} \quad \ldots\ldots\ldots (7)\ \&\ (7A)$$

$$AC : BC :: Ab : bh$$
$$bc : ac :: bh : aB$$
$$AC\times bc : BC\times ac :: Ab\times bh : aB\times bh$$
$$AC\cdot bc\cdot aB = BC\cdot ac\cdot Ab$$

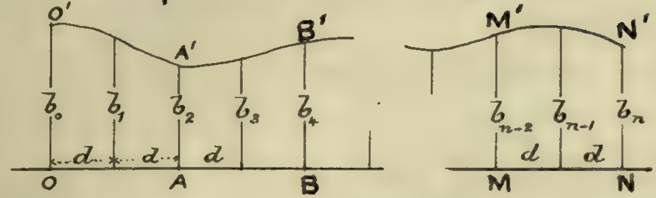
but $ac = ab + bc$

$$\therefore \quad bc\cdot aB\cdot AC = ab\cdot BC\cdot Ab + bc\,BC\cdot Ab$$

$$\therefore \quad bc = \frac{ab\cdot BC\cdot Ab}{aB\cdot AC - BC\cdot Ab} \quad \ldots\ldots\ldots\ldots (8)$$

n.33
§32

Simpson's Rule.



The most general equation of the parobola is

$$y = A + Bx + Cx^2.$$

For the area of OO'A'A

$$Area = \int_0^{2d} y\,dx = \int_0^{2d}(A\,dx + Bx\,dx + Cx^2 dx)$$

$$Area = 2Ad + 2Bd^2 + \frac{8}{3}Cd^3;$$

to determine A, B and C the equation of the curve gives

$$y_0 = b_0 = A$$
$$y_1 = b_1 = A + Bd + Cd^2$$
$$y_3 = b_3 = A + 2Bd + 4Cd^2;$$

solving these for A, B, and C we have, $A = b_0$

$$B = \frac{-3b_0 + 4b_1 - b_2}{2d}$$
$$C = \frac{2b_0 - 4b_1 + 2b_2}{4d^2} \quad \therefore$$

$$Area = 2b_0d - 3b_0d + 4b_1d - b_2d + \tfrac{4}{3}b_0d - \tfrac{8}{3}b_1d + \tfrac{4}{3}b_2d$$
$$= \tfrac{1}{3}d[6b_0 - 9b_0 + 12b_1 - 3b_2 + 4b_0 - 8b_1 + 4b_2],$$
$$= \tfrac{1}{3}d(b_0 + 4b_1 + b_2).$$

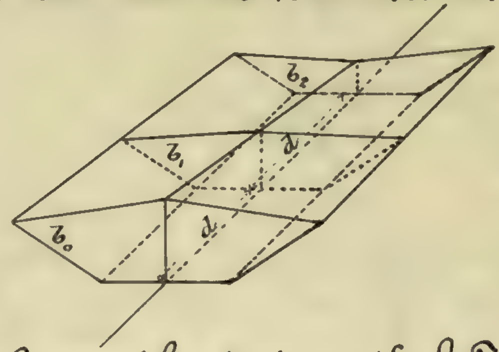
$$Area\ AA'B'B = \tfrac{1}{3}d[b_2 + 4b_3 + b_4]$$

$$Area\ MM'N'N = \tfrac{1}{3}d[b_{n-2} + 4b_{n-1} + b_n];$$

hence the total area will be

$$= [b_0 + b_n + 2(b_2 + b_4 \cdots\cdots) + 4(b_1 + b_3 \cdots\cdots)]\frac{d}{3} \quad \text{(6)}$$

Prismoidal Formula.



Since the Prismoidal Formula is identical with Simpson's Rule it is introduced here.
It is used for computing the volume of an irregular solid when we know the end and middle areas, the X sections being parallel

The solid is supposed to be bounded by planes which are continuous from one end area to the other.

Such a solid may be divided into pyramids and wedges.

For any pyramid

$$Area = A_p = m(x+b)^2$$

where b is the distance from the origen to the vertex, & m = constant

For any wedge

$$Area = A_w = m'(x+b)';$$

then the totol area = A

$$A = \Sigma A_p + \Sigma A_w = \Sigma m(x+b)^2 + \Sigma m'(x+b')$$

Collecting the coefficients

$$A = D + Bx + Cy$$

$$Volume = \int_0^{2d} A\,dx = 2Dd + 2Bd^2 + \frac{8}{3}Cd^3.$$

Similarly $V = \frac{1}{3}d[b_0 + 4b_1 + b_2]$

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Amstlers Planimeter

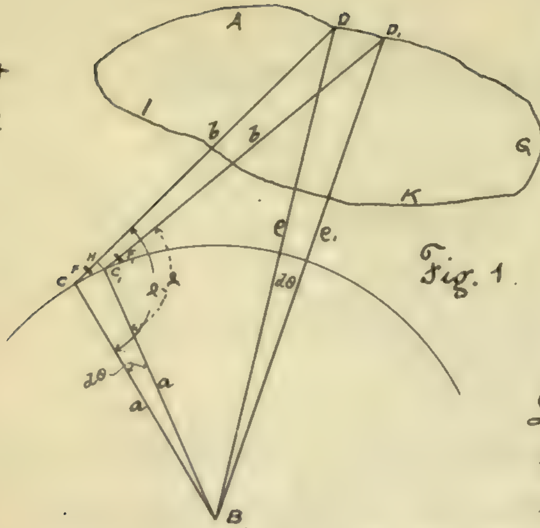


Fig. 1

AGKI = Area to be measured
C = joint
D = tracing point
F = wheel
A = Area of AGKI
BC = a, constant
CD = b, "

Let the tracing point be moved from D to D_1 an infinitessimal distance; then by polar coördinates we have

$$\text{Area } DBD_1 = dA = \tfrac{1}{2}\rho^2 d\theta \quad \text{or generally}$$

$$A = \tfrac{1}{2}\int \rho^2 d\theta \quad \ldots\ldots 1$$

From the triangle BCD we have

$$\rho^2 = a^2 + b^2 - 2ab\cos\alpha \quad \ldots\ldots 2$$

$$\therefore \quad A = \frac{a^2+b^2}{2}\int d\theta - ab\int \cos\alpha\, d\theta \quad \ldots\ldots 3$$

Since in measuring the small area $DBD_1 = \frac{1}{2}\rho^2 d\theta$ we supposed ρ constant, α, which varies with ρ, may also be supposed constant during the same movement; this gives

$$\text{angle } CBC_1 = \text{angle } DBD_1 = d\theta \quad \therefore \; CC_1 = a\,d\theta$$

$$CH = CC_1 \sin DCC_1 = a\cos\alpha\, d\theta;$$

but CH is the component of the motion of the point C which is perpendicular to the line CD and is measured by the rotation of the wheel F, since it slides in the direction of its axis CD; hence we see the integration of the second term of Eq. 3 is performed by the wheel F and is equal to $b \times$ the distance rolled; this supposes the wheel to be at the joint C, but it is evident that the distance rolled at the point F, always very near C, will be in proportion to what it would be if the wheel were actually at the joint. The wheel F is usually divided into 100 parts while the distance CD = b can be changed so as to read square inches or any other unit of area.

If the pole B is outside the figure to be measured as in Fig. 1 then we integrate between the limits 0 and 0. Eq. 3 becomes,

$$A = \frac{a^2+b^2}{2}\int_0^0 d\theta - ab\int_0^0 \cos\alpha\, d\theta$$

The first term reduces to 0 and the second is read on the wheel.

p.34 §32 Cont. If the pole is inside the area to be measured we have

$$A = \frac{a^2+b^2}{2}\int_0^{2\pi} d\theta - ab\int_0^{2\pi}\cos\alpha\, d\theta .$$

Let $a^2+b^2=r^2$ then $A=\pi r^2 + b \times$ distance rolled by the wheel F.
If α is kept constant and $= 90^\circ = \frac{\pi}{2}$, then $\cos\alpha = 0$ and $\therefore$

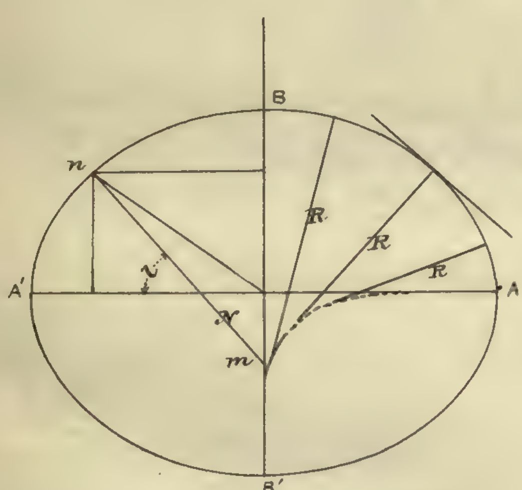
$$A = \pi r^2 ;$$

this is called the corrective circle and must always be added to the reading of the wheel when the pole is inside the figure.

Amstler's is the simplest, cheapest and most generally used of planimiters, though the Roller Planimeter at 3 times the cost has, it is claimed, 6 or 7 times the accuracy.

Amstler's is made by Buff & Berger of Boston and others and the Roller by Fauth & Co. 132 Md. Av., Washington D. C.

p.46 §33 note



It will be convenient first to recall the expressions of several lines of an ellipse in terms involving the latitude, l, which is the angle that the normal to any point on the ellipse makes with the major axis.

Designating the equatorial semiaxis by a, the minor or polar semi-axis by b, then the eccentricity e is expressed by

$$e^2 = \frac{a^2-b^2}{a^2}$$

The normal nm by, $N = \dfrac{a}{(1-e^2\sin^2 l)^{1/2}}$

The radius of curvature by, $R = \dfrac{a(1-e^2)}{(1-e^2\sin l)^{3/2}}$

l = latitude.

p.46 m = length in feet of 1′ of latitude

§ 33 m_1 = " " " " 1′ " " at 45°.

*note We have $m = \frac{2\pi R}{360 \times 60} = \frac{\pi R}{180 \times 60}$

Smit. Replacing R by its value.

$$m = \frac{\pi a}{180 \times 60}(1-e^2)(1-e^2\sin^2 l)^{-\frac{3}{2}}$$

Developing the last factor of this by the binomial theorem it may be written

$$m = \frac{\pi a}{60 \times 180}(1-e^2)\left(1+\tfrac{3}{2}e^2\sin^2 l + \tfrac{15}{8}e^4\sin^4 l + \ldots\ldots\right)$$

Since for the earth e is very small (= 0.082271, Clarke) its fourth power may be neglected in comparison with the other terms and the above equation may be written

$$m = O + P\sin^2 l, \text{ where } \begin{cases} O = \frac{\pi a}{60 \times 180}(1-e^2) \\ P = \frac{\pi a}{60 \times 180}(1-e^2)\frac{3}{2}e^2 \end{cases}$$

Let m_1 be the value for m when $l = l' = 45°$

$$m_1 = O + P\sin^2 l'$$

$$m - m_1 = P(\sin^2 l - \sin^2 l')$$

$$= P\{\sin(l+l')\sin(l-l')\}$$

let $l - l' = \pm\delta$ then

$$m - m_1 = P\sin(2l' \pm \delta)\sin\delta$$

$$= P(\sin 2l'\cos\delta \pm \cos 2l'\sin\delta)\sin\delta$$

but $l' = 45°$ ∴ $\sin 2l' = 1$, and $\cos 2l' = 0$

$$\therefore m - m_1 = P\sin\delta\cos\delta = \frac{P}{2}\sin 2\delta$$

$$\therefore m = m_1\left(1 - \frac{P}{2m_1}\sin 2\delta\right)$$

$$\frac{P}{2m_1} = \frac{1}{2 \times 6076.36} \times \frac{\pi a}{60 \times 180}(1 - 0.082271^2) \times \tfrac{3}{2}(0.082271)^2$$

Numerator			Denominator		
π = 3.14159265	log.	0.4971499	12152.72	log	4.0846735
a = 20923831	"	7.3206413	10800	"	4.0334238
$(1-e^2)$ = 0.99323	"	9.9970498 − 10	2	"	0.3010300
3		0.4771213			− 8.4191273
e^2 = 0.006768		7.8304604 − 10			+ 6.1224227
		+ 6.1224227	$\frac{1}{198.02}$	"	− 2.2967[illegible]

$$\therefore \frac{P}{2m'} = \frac{1}{198.02} = \frac{1}{200} \text{ nearly}$$

$$\therefore m = 6076.36\left(1 \pm \frac{\sin 2\delta}{200}\right) \ldots\ldots\ldots\ldots (1)$$

We can check this value of $\frac{p}{2m_1}$ by using the length of 1′ as given in the U.S. Coast Survey projection tables, we then have

$$\overset{m \text{ for } 35^\circ}{6065.37} = \overset{m_1 \text{ for } 45^\circ}{6075.87} - \frac{p}{2m_1}\sin 20^\circ \times \overset{m_1 \text{ for } 45^\circ}{6075.87}$$

where $\delta = -10^\circ$; this gives for $\frac{p}{2m_1}$ the value $\frac{1}{201.05}$.

To find the value of m' = length in feet of 1′ on a great circle perpendicular to the meridian we first find its value at $l = 45^\circ$. The radius of curvature of this great circle is the normal $N = \frac{a}{(1-e^2\sin^2 l)^{1/2}}$, this is evident since the surface is one of revolution its centre must be in the axis BB′.

But $\qquad m' : m :: N : R$

$$\therefore \frac{m'}{m} = \frac{N}{R} = \frac{\frac{a}{(1-e^2\sin^2 l)^{1/2}}}{\frac{a(1-e^2)}{(1-e^2\sin^2 l)^{3/2}}} = \frac{1-e^2\sin^2 l}{1-e^2}$$

If m_1' and m_1 be the values when $l = 45^\circ$ then

$$m_1' = m_1 \frac{1-e^2\sin^2 45^\circ}{1-e^2} = m_1 \frac{1-0.003384}{1-0.006768} = m_1 \frac{0.996616}{0.993232}$$

or $\quad m_1' = m_1\left(1 + \frac{3384}{993232}\right) = m_1\left(1+\frac{1}{300}\right)$ nearly

To find the general value for m' we have

$$m' = \frac{2\pi N}{360 \times 60} = \frac{\pi a}{60 \times 180}(1-e^2\sin^2 l)^{-1/2}$$

developing by the binomial theorem

$$m' = \frac{\pi a}{10800}\left(1 + \frac{e^2}{2}\sin^2 l + \tfrac{3}{8}e^4\sin^4 l - \cdots\cdots\right)$$

dropping as before the higher powers of e we have,

$$m' = O' + P'\sin^2 l \; ; \quad \text{where} \begin{cases} O = \frac{\pi a}{10800} \\ P = \frac{\pi a}{10800}\cdot\frac{e^2}{2} \end{cases}$$

In the same way that we found the value of m we find

$$m' = m_1' - \frac{P}{2}\sin 2\delta$$

but $\quad m_1' = m_1\left(1+\frac{1}{300}\right) \; \therefore$

$$m' = m_1\left(1 + \frac{1}{300} - \frac{P}{2m_1}\sin 2\delta\right)$$

Substituting their values we have

$$\frac{P}{2m_1} = \frac{\pi a e^2}{2 \times 10800 \times 2 \times 6076.36} = \frac{1}{590.03}$$

$$\therefore m' = 6076.36\left(1 + \frac{1}{300} - \frac{\sin 2\delta}{600}\right) \cdots\cdots\cdots\cdots (2)$$

The ± sign is not necessary if we observe that $\sin 2\delta$ ordinarily has the same sign as δ.

p.46 To find the value of m'' the length of 1′ on a great circle
§ 33 which makes the angle θ with the meridian we have
*note From Courtnay's Calculus p. 240 § 208
Cont.

$$R = \frac{R_1 R_2}{R_2 \cos^2\theta + R_1 \sin^2\theta},$$

where R_1 and R_2 are the least and greatest radii of curvature.
∴ Since the arc subtending 1′ varies as the radius we have

$$m'' = \frac{m m'}{m \cos^2\theta + m' \sin^2\theta}$$

which may be substituted for Rankine's Equation (3) when $m > m'$

The other equations following viz. (4), (5) and (6) give no trouble in obtaining them; the last is readily seen from the figure given in these notes

For values of m and m' see U.S. Coast Survey, Report 1884, Appendix No. 6,

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PROBLEM 2ND.

p.48 $\frac{a}{r} = \alpha$; $\sin\alpha = \sin\frac{a}{r}$; developing by Maclaurin's Theorem,
§ 33

$$\sin\alpha = \frac{a}{r} - \frac{1}{1\cdot2\cdot3}\frac{a^3}{r^3} + \frac{1}{1\cdot2\cdot3\cdot4\cdot5}\frac{a^5}{r^5} - \text{etc} \ldots \qquad (53)$$

$$\therefore \frac{a}{r} = \sin\alpha + \frac{1}{1\cdot2\cdot3}\frac{a^3}{r^3} - \frac{1}{1\cdot2\cdot3\cdot4\cdot5}\frac{a^5}{r^5} + \text{etc} \ldots$$

since α is very small $\frac{a}{r} = \sin\alpha$, nearly; substituting this value in the above we get the following approximation,

$$\frac{a}{r} = \sin\alpha + \frac{1}{1\cdot2\cdot3}\sin^3\alpha - \frac{1}{1\cdot2\cdot3\cdot4\cdot5}\sin^5\alpha + \text{etc} \ldots \qquad (54)$$

To get equations (55A) and (56A) we have only to recollect that $\log(1+x) = M\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc}\ldots\right)$ where M is the modulus of the system, in the common system $M = .4342945$ $\therefore \frac{M}{6} = .0723824$

PROBLEM 3RD The usual formula for the area of a spherical triangle is $S = \frac{1}{2}(A + B + C - 2)\pi r^2$
where A, B and C have for a unit of measure one right angle. Using degrees as the unit we must multiply and divide by 90, we then have $S = \frac{1}{2}\frac{(A+B+C-2)90}{90}\pi r^2$
but $(A+B+C-2)90 = X$ = spherical excess

$$\therefore \quad S = \frac{X\pi r^2}{180} \text{ and } X = 360\frac{S}{2\pi r^2} \text{ -----} (57)$$

PROBLEM V 1st METHOD

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§ 33

Assumed from trigonometry
$$\begin{cases} \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ 2\sin^2\frac{\gamma}{2} = 1-\cos\gamma \\ 2\cos^2\frac{\gamma}{2} = 1+\cos\gamma \end{cases}$$

$$\cos\gamma = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos C \quad \ldots\ldots (61)$$

but $2\sin^2\frac{\gamma}{2} = 1-\cos\gamma$

$$= 1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos C$$

adding and subtracting $\sin\alpha\sin\beta$

$$= 1 - \cos\alpha\cos\beta + \sin\alpha\sin\beta - \sin\alpha\sin\beta - \sin\alpha\sin\beta\cos C$$

$$= 1 - \cos(\alpha+\beta) - (1+\cos C)\sin\alpha\sin\beta$$

$$= 1 - \cos(\alpha+\beta) - 2\cos^2\frac{C}{2}\sin\alpha\sin\beta$$

$$= 2\sin^2\frac{\alpha+\beta}{2} - 2\cos^2\frac{C}{2}\sin\alpha\sin\beta$$

dividing by 2 and making $\sin D = \frac{C}{2}\sqrt{\sin\alpha\sin\beta}$ we have

$$\sin^2\frac{\gamma}{2} = \sin^2\frac{\alpha+\beta}{2} - \sin^2 D\,; \quad \text{since } \sin^2\frac{\alpha+\beta}{2} + \cos^2\frac{\alpha+\beta}{2} = 1$$

we may write $\sin^2\frac{\gamma}{2} = \sin^2\frac{\alpha+\beta}{2} - \sin^2\frac{\alpha+\beta}{2}\sin^2 D - \cos^2\frac{\alpha+\beta}{2}\sin^2 D$

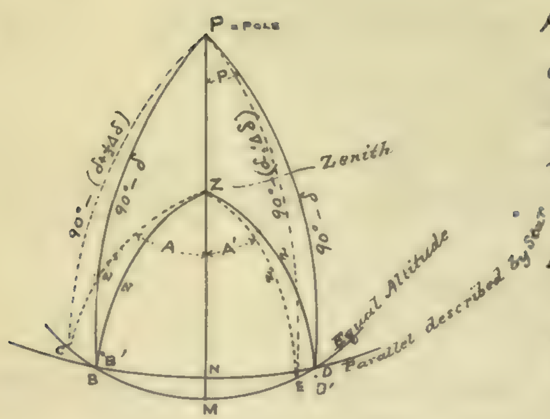
$$= \sin^2\frac{\alpha+\beta}{2}\cos^2 D - \cos^2\frac{\alpha+\beta}{2}\sin^2 D$$

$$= \left(\sin\frac{\alpha+\beta}{2}\cos D + \cos\frac{\alpha+\beta}{2}\sin D\right)\left(\sin\frac{\alpha+\beta}{2}\cos D - \cos\frac{\alpha+\beta}{2}\sin D\right)$$

$$= \sin\left(\frac{\alpha+\beta}{2}+D\right)\sin\left(\frac{\alpha+\beta}{2}-D\right)$$

$$\therefore \sin\frac{\gamma}{2} = \sqrt{\left\{\sin\left(\frac{\alpha+\beta}{2}+D\right)\sin\left(\frac{\alpha+\beta}{2}-D\right)\right\}}\,; \quad \ldots\ldots\ldots (62)$$

p 72
§42



A' = east azimuth at 1st observation
A = west " " 2nd "
φ = latitude
δ = suns declination at noon
h = altitude of sun when observed
z = zenith distance = $90° - h$
$\Delta\delta = 2DD' = 2BB'$ = increase of declination between 1st and 2nd obs.

Let P be the pole, Z the zenith ANB the parallel described by a star in its diurnal motion

AMB a small circle parallel to the horizon, and PZM the meridian of the place.

From the triangle PZE we have (see R.C.E p.49 Eq. 61)

$$\cos PE = \cos ZP \cos z + \sin ZP \sin z \cos PZE$$

or $$\sin(\delta - \tfrac{1}{2}\Delta\delta) = \sin\varphi \sin h - \cos\varphi \cos h \cos A'$$

similarly $$\sin(\delta + \tfrac{1}{2}\Delta\delta) = \sin\varphi \sin h - \cos\varphi \cos h \cos A$$

the difference of which gives, (see R.C.E. p.41 Eq. 13, 14, & 21)

$$2 \cos\delta \sin\tfrac{1}{2}\Delta\delta = 2\cos\varphi \cos h \sin\tfrac{1}{2}(A + A') \sin\tfrac{1}{2}(A - A')$$

since A and A' are nearly equal we have $\sin\frac{1}{2}(A - A') = \frac{A - A'}{2}$ and $\sin\frac{1}{2}(A + A') = \sin A$, also since $\Delta\delta$ is but a few minutes $\sin\frac{1}{2}\Delta\delta = \frac{1}{2}\Delta\delta$. therefore we have with sufficient accuracy

$$A - A' = \frac{\Delta\delta \cos\delta}{\cos\varphi \cos h \sin A}$$

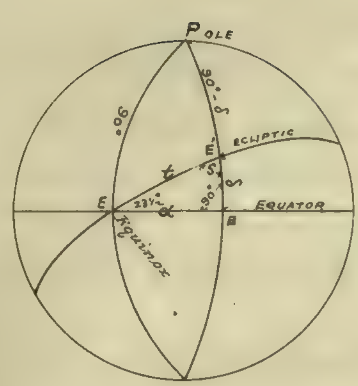
It will be necessary to take the times of observation in order to determine $\Delta\delta$. If we take half the elapsed time as the hour angle P, then from the triangle APZ

$$\sin Z : \sin P :: \sin PD : \sin ZD$$

$$\therefore \quad \sin A = \frac{\sin P \cos\delta}{\cos h}$$

$$\therefore (A - A') = \frac{\Delta\delta}{\cos\varphi \sin P}$$

$$\therefore \quad \tfrac{1}{2}(A - A') = \frac{\Delta\delta}{2} \sec\varphi \operatorname{cosec} P \quad \ldots\ldots\ldots (1)$$



To determine the rate of change of the suns declination

δ = declination

α = right ascension

t = mean time

$\mathcal{E}$ = inclination of the Ecliptic = 23½°

S = angle between any meridian and the Ecliptic.

At the equinox $S = 90° - 23\frac{1}{2} = 66\frac{1}{2}°$

From the figure it may be seen that

$$\frac{d\delta}{dt} = \cos S$$

But from the right angled triangle $\mathcal{E}\mathcal{E}'B$ we have by Napier's Rule

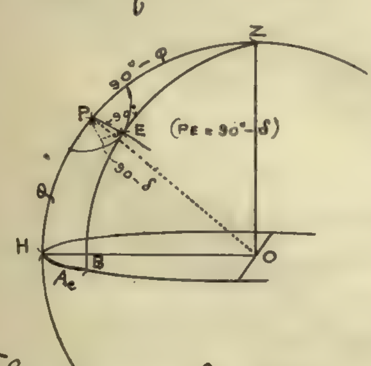
$$\cos. S = \sin \mathcal{E} \cos \alpha$$

$$\therefore \frac{d\delta}{dt} = \sin \mathcal{E} \cos \alpha$$

At the equinox $\alpha = 0 \therefore d\delta = dt \sin \mathcal{E}$. In one year the sun travels 360°, and taking the distance traveled in one hour $= dt$ we have
$$dt = \frac{360°}{365\frac{1}{4} \times 24} = 0°02'27''.6 \quad \text{or} \quad 147''.6$$

$$\therefore \text{at the equinox } d\delta = 147''.6 \times \sin 23\frac{1}{2}° = 59''$$

Since $\sin \mathcal{E}$ is constant we see that $d\delta$ varies as $\cos \alpha$. These formulæ would be exact if the motion of the sun was uniform, which owing to the earth's orbit being elliptical is not quite true.



φ = latitude of place of observation

δ = declination of star

$P\mathcal{E}$ = polar distance = $90° - \delta$

$PZ\mathcal{E} = HB = A_\varepsilon$ = Azimuth at elongation

P is the pole, Z zenith, H horizon

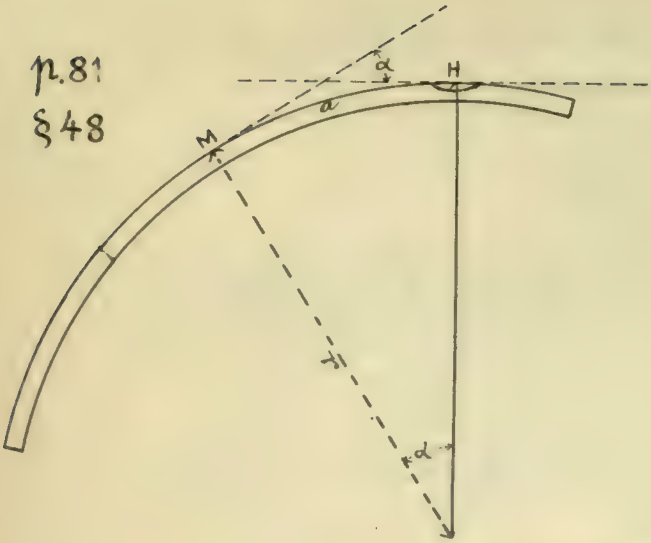
The triangle $PZ\mathcal{E}$ is right angled at $\mathcal{E}$ so by Napier's Rule

$$\sin P\mathcal{E} = \cos.\text{comp. } PZ \cos.\text{comp. } PZ\mathcal{E}$$

$$\text{or} \quad \cos\delta = \cos\varphi \sin A_\varepsilon$$

$$\therefore \sin A_\varepsilon = \frac{\cos\delta}{\cos\varphi} \text{ ---------- } (2)$$

p.81
§48



M = middle of tube,
H = highest point of tube;
arc equal to radius expressed in seconds = 206264.8,

$$\alpha'' : 206264''.8 :: a : r,$$

$$\alpha = 206264''.8 \frac{a}{r} \quad \text{......} \quad (1)$$

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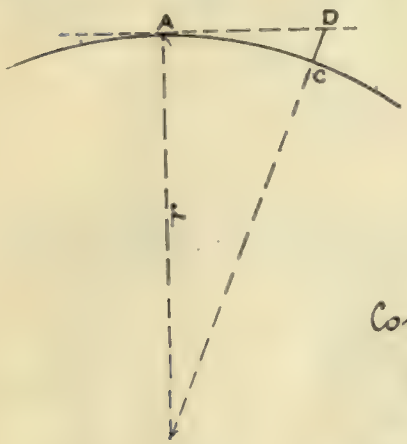
p.88
§52



$2r$ = mean diameter of the Earth
= 7912½ miles = 41,778,000 feet.

$$AD^2 = DC(DC + 2r)$$

$$\therefore \quad DC = \frac{AC^2}{2r} \text{ nearly}$$

Correction, for curvature in feet $= \dfrac{\text{distance}^2}{41,778,000}$ .. (2)

For the distance one mile,

correction in ft. $= \dfrac{(5280)^2}{7912\tfrac{1}{2} \times 5280} = \dfrac{5280}{7912\tfrac{1}{2}} = \dfrac{2}{3}'$ nearly

∴ Cor. in ft. $= \frac{2}{3}$ (distance in miles)^2 .... (2)

$HAC = H'BD = \frac{1}{2}\theta$;
$BAB' = ABA'$ nearly
$CAB = ABD$; $\alpha = \beta$

$$\left.\begin{aligned} \alpha &= CAB = HAB' + CAH - BAB' \\ \beta &= ABD = H'BA' - H'BD + ABA' \end{aligned}\right\} \therefore$$

$$\frac{\alpha+\beta}{2} = \frac{2\alpha}{2} = \frac{2\beta}{2} = \alpha = \beta = \frac{HAB' + H'BA'}{2} =$$

$$\frac{\alpha' + \beta'}{2} \text{ nearly}$$

$$\therefore \quad \alpha = \beta = \frac{\alpha' + \beta'}{2} \text{ nearly}$$

Levelling by the Barometer and Thermometer

p.92 §57 Supposing the air to be at rest, that is no wind blowing, then the only force acting is gravity $= -g$, which acts in the direction of the axis of Z, and since the pressure p varies with the density D we have

$$dp = -g D dz = \text{weight of vol. of air } 1 \times 1 \times dz$$

but by Marriott's Law $\quad p = p_0 D$, dividing by this.

$$\frac{dp}{p} = -g\frac{dz}{p_0}.$$

Integrating $\quad \log p = -\frac{gz}{p_0} + \log C \quad \therefore$

$$p = C \varepsilon^{-\frac{gz}{p_0}}$$

Making $z = 0$ and denoting the corresponding pressure by p_1 we have $\quad p_1 = C \quad \therefore \frac{p}{p_1} = \varepsilon^{-\frac{gz}{p_0}}$

$$\therefore \log \frac{p}{p_1} = -\frac{gz}{p_0} \log \varepsilon = -M \frac{gz}{p_0}$$

Where M is the modulus of the common system,

$$\therefore \quad z = \frac{p_0}{Mg} \log \frac{p_1}{p} \quad \text{.........(A)}$$

If the temperature of the mercury had not changed we would have $\quad \frac{p_1}{p} = \frac{H}{h'} \quad$ (h' = height of mercury at upper station)

Reducing the barometric column h to what it would have been had the temperature of the mercury at the upper not differed from that at the lower; the coëfficient of expansion for mercury being 0.0001001 ;

$$\frac{p_1}{p} = \frac{H}{h(1 + (T - t')\, 0.0001001)}$$

From the equation for permanent gases when the temperature varies $\quad p = p_0 D(1 + \alpha\theta)$

making $\theta = 0$ $\quad p = p_0 D_1 \quad$ and dividing the first by the second

$$1 = \frac{D}{D_1}(1 + \alpha\theta) \quad \therefore D = \frac{D_1}{1 + \alpha\theta}$$

But no matter what the temperature may be,

$$p = p_0 D = \frac{p_0 D_1}{1 + \alpha\theta}$$

or $\quad \frac{p}{D_1} = \frac{p_0}{1 + \alpha\theta} = \frac{D_m h_{,,}\, g}{D_1} = $ a constant (see next page).

p.92 §57 Cont. In this equation D_m is the density of mercury at standard temperature (32°F.); g is the force of gravity which, though it varies slightly with the latitude, Rankine considers constant; h_{11} is the height of the column of mercury at standard temperature (32°F.) which varies as D_1 the density at standard temperature (32°F.); consequently $\frac{h_{11}}{D_1}$ is constant. $\therefore$ the last term in the above equation is a constant.

It is evident that owing to atmospheric changes the pressure varies, for the same temperature, from day to day and this is shown by the varying height of the barometric column h_{11} and the varying density of the air D_1.

Since the coefficient of expansion for permanent gases $\alpha = \frac{1}{491}$, and $\theta = t_1 - 32$ we have from the last equation

$$p_0 = \frac{D_m h_{11}}{D_1} g \left(1 + \frac{t_1 - 32}{491}\right)$$

Substituting this and the value already found for $\frac{p_1}{p}$ in Eq.(A) we have

$$z = \frac{D_m h_{11}}{\mathfrak{M} D_1}\left\{\log H - \log h\left(1 + (T - t)\,0.0001001\right)\right\}\left(1 + \frac{t_1 - 32}{491}\right) \quad \ldots (B)$$

In this equation t_1 is the temperature of the air, but as this may not be, indeed scarcely ever is, the same at both stations a difficulty arises in applying the formula. Let X denote the factor in the second member into which the factor involving t_1 is multiplied. If T' is the temperature at the lower station and it be supposed the same throughout to the upper station, then will

$$z_1 = \left(1 + \frac{T' - 32}{491}\right) X$$

If t' be the temp. at the upper station and it be assumed the same for both;

$$z_2 = \left(1 + \frac{t' - 32}{491}\right) X$$

The mean of these is the best approximation we can make

$$\frac{z_1 + z_2}{2} = z = \left(1 + \frac{T' + t' - 64}{982}\right) X$$

$$\therefore \quad z = \frac{D_m h_{11}}{\mathfrak{M} D_1}\left\{\log H - \log h - \log\left(1 + (T - t)\,0.0001001\right)\right\}\left(1 + \frac{T' + t' - 64}{982}\right)$$

But $-\log\left(1 + (T - t)\,0.000101\right) = -.00044(T - t)$ nearly

$$\left\{\begin{array}{l} \text{since when } T - t = 1,\ \log(1 + 0.0001001) = 0.0000434435 \\ \qquad\quad " \quad T - t = 10,\ \log(1 + 0.001001) = 0.0004341 \\ \qquad\quad " \quad T - t = 100,\ \log(1 + 0.01001) = 0.004314 \end{array}\right\}$$

$\therefore$ for values of $T - t$ less than 100 this approximation is close enough

p.92 substituting this value and using Rankines value $\frac{1}{986}$
§57 for the double coefficient of the expansion of air.
Contd.

$$z = \frac{D_m h_{ii}}{\mathfrak{M} D_i}\left\{\log H - \log h - 0.000044(\mathfrak{T}-t)\right\}\left(1+\frac{\mathfrak{T}'+t'-64}{986}\right)$$

We have seen that $\frac{D_m h_{ii}}{\mathfrak{M} D_i}$ is a constant, we can therefore determine its value by measuring some one z with the level or theodolite, the other quantities H, h, $\mathfrak{T}$, t, $\mathfrak{T}'$, t' can all be observed. Bartlett who has been mainly followed in the above demonstration gives for this constant 60345.51 while Rankine finds it to be 60360, substituting this in the above

$$z = 60360\left\{\log H - \log h - 0.000044(\mathfrak{T}-t)\right\}\left(1+\frac{\mathfrak{T}'+t'-64}{986}\right) \cdots\cdots (1)$$

Rankine's Equation (2) is only approximate and the value of the constant may be determined by computing z by the longer formula Eq.(1) when all the quantities in Eq.(2) become known excepting the constant.

When no logarithms are at hand we have, since the coefficient of expansion of mercury $= \frac{1}{10000}$

$$h' = h\left(1+\frac{\mathfrak{T}-t}{10000}\right)$$

$$\log\frac{H}{h'} = \log\left(1+\frac{H-h'}{h'}\right),$$

developing by Taylor's Theorem,

$$\log\frac{H}{h'} = 0 + \mathfrak{M}\left(\frac{H-h'}{h'} - \frac{(H-h')^2}{1\cdot 2\, h'^2} + \frac{(H-h')^3}{1\cdot 2\cdot 3\, h'^3} - \text{etc} \cdots\right).$$

When H and h' do not differ much we may neglect all but the first term and consider $\frac{H+h'}{2} = h'$

$$\therefore \log\frac{H}{h'} = 2\mathfrak{M}\,\frac{H-h'}{H+h'} \cdots \left\{\begin{array}{lr} \mathfrak{M} = 0.4342945 \quad \log & 9.6377843-10 \\ \times 2 \qquad\qquad\qquad " & .3010300 \\ \times 60360 & 4.7807492 \\ = 52428 & 4.7195635 \end{array}\right.$$

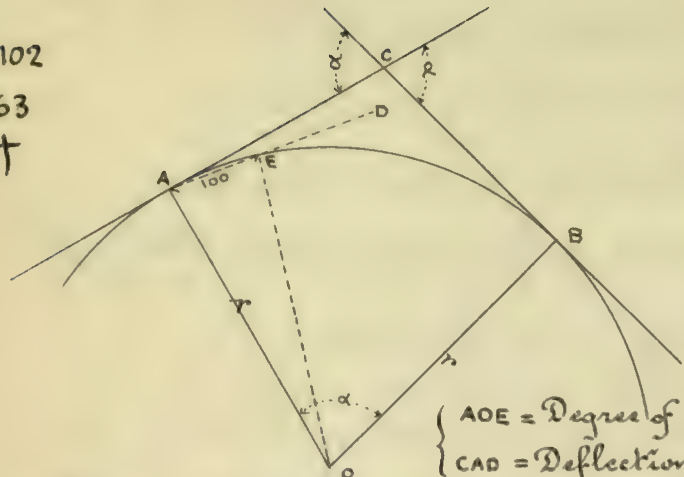
$$\therefore z = 52428\,\frac{H-h'}{H+h'}\left(1+\frac{\mathfrak{T}'+t'-64}{986}\right) \cdots (3) \text{ nearly.}$$

Note H at sea level = 30.00 in.
h' " 3000 ft. above sea level = 26.75 } nearly; and $\frac{H-h'}{h'} = \frac{1}{9}$ nearly

then $\log\frac{H}{h'} = \mathfrak{M}\left(\frac{1}{9} - \frac{1}{162} + \frac{1}{4374} - \text{etc}\right)$

$52428 \times \frac{1}{2} \times \frac{1}{162} = 162$ hence we see that neglecting the second term in the development as is done in equation (3) gives an erroneous result by 160 ft. about.

p.102
§63
†



The <u>deflection angle</u> as usually used by Americans is the angle between the tangent and the chord = $\frac{1}{2}$ the angle at the centre subtended by an arc 100 ft. long

$$\left\{\begin{array}{l} AOE = \text{Degree of curvature} \\ CAD = \text{Deflection Angle} \end{array}\right\}$$

For an 1° curve, $2\pi r_1 = 360 \times 100 = 36000$ $\quad \therefore r_1 = 5729.6$ ft.

" a 2° " $2\pi r_2 = \frac{360}{2} \times 100 = \frac{36000}{2}$ $\quad \therefore r_2 = \frac{5729.6}{2}$

" " $n°$ " $2\pi r_n = \frac{360}{n} \times 100 = \frac{36000}{n}$ $\quad \therefore r_n = \frac{5729.6}{n}$

$\therefore \; n = \frac{5729.6}{r}$ = Degree of curvature = twice Deflection angle as ordinarily used.

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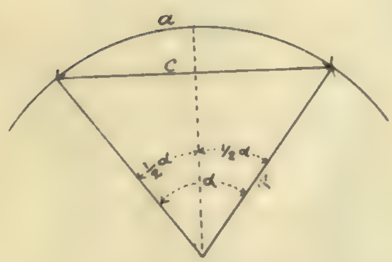
p.104
§63



$$\alpha = \frac{a}{r}.$$

$$c = 2r \sin\tfrac{1}{2}\alpha = 2r \sin\frac{a}{2r}.$$

Developing by Maclaurin's Theorem

$$c = 2r\left(\frac{a}{2r} - \frac{a^3}{1\cdot2\cdot3(2r)^3} + \frac{a^5}{1\cdot2\cdot3\cdot4\cdot5(2r)^5} - \text{etc.}\right)$$

$$= a - \frac{a^3}{24r^2} \quad \text{nearly}$$

$$\therefore \quad c = a\left(1 - \frac{a^2}{24r^2}\right) \text{ nearly} \; \ldots\ldots \; (5)$$

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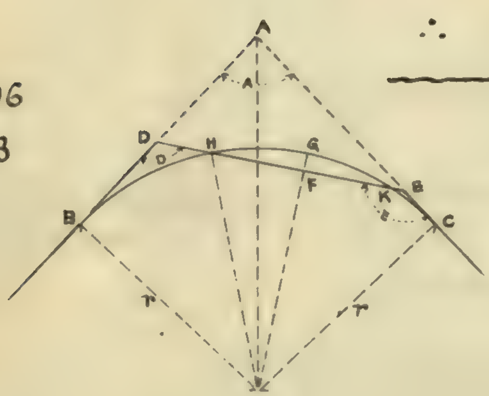
p.106
§ 63



$$\left.\begin{array}{l} BD^2 = DK\cdot DH = (DF+FH)(DF-FH) = DF^2 - FH^2 \\ EC^2 = EH\cdot EK = (EF+FH)(EF-FH) = EF^2 - FH^2 \end{array}\right\} \therefore$$

$$BD^2 - EC^2 = DF^2 - EF^2 = (DF+EF)(DF-EF)$$

$$= DE\left\{DF - (DE-DF)\right\} = DE(2DF - DE) \therefore$$

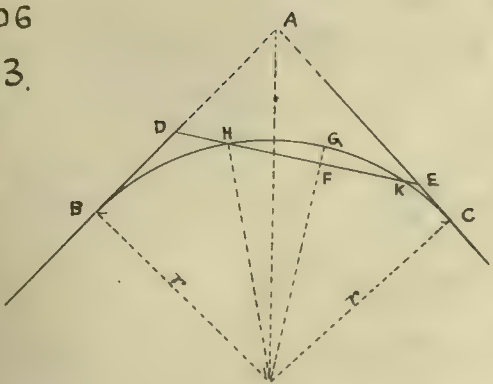
$$DF = \frac{DE}{2} + \frac{BD^2 - EC^2}{2DE} \; \ldots\ldots\ldots\ldots \; (9)$$

$$EF = DE - DF = DE - \left\{\frac{DE}{2} + \frac{BD^2 - EC^2}{2DE}\right\} \therefore$$

$$EF = \frac{DE}{2} - \frac{BD^2 - EC^2}{2DE} \; \ldots\ldots\ldots\ldots \; (9)$$

(over)

p.106
§63.
Cont.



$$\left.\begin{array}{l} BD^2 = DF^2 - FH^2 \\ EC^2 = EF^2 - FH^2 \end{array}\right\} \therefore \quad BD^2 + EC^2 = DF^2 + EF^2 - 2FH^2 \quad \therefore$$

$$2FH^2 = DF^2 + EF^2 - (BD^2 + EC^2) \text{ ......}(A)$$

$$2FH^2 = \left(\frac{DE}{2} + \frac{BD^2 - EC^2}{2DE}\right)^2 + \left(\frac{DE}{2} - \frac{BD^2 - EC^2}{2DE}\right)^2 - (BD^2 + EC^2)$$

$$2FH^2 = \left\{\begin{array}{l} \frac{DE^2}{4} + \frac{BD^2 - EC^2}{2} + \frac{(BD^2 - EC^2)^2}{4DE^2} + \\ \frac{DE^2}{4} - \frac{BD^2 - EC^2}{2} + \frac{(BD^2 - EC^2)^2}{4DE^2} \end{array}\right\} - (BD^2 + EC^2)$$

$$\therefore \quad 2FH^2 = \frac{DE^2}{2} + \frac{(BD^2 - EC^2)^2}{2DE^2} - (BD^2 + EC^2)$$

$$FH = \sqrt{\left\{\frac{DE^2}{4} + \frac{(BD^2 - EC^2)^2}{4DE^2} - \frac{BD^2 + EC^2}{2}\right\}} \text{ ....}(10)$$

From equation (A) above, we have

$$FH = \frac{\sqrt{\{DF^2 + EF^2 - BD^2 - EC^2\}}}{\sqrt{2}} \text{ ....}(10)$$

$$\left.\begin{array}{l} DK = \frac{DK + DH}{2} + \frac{DK - DH}{2} \\ DH = \frac{DK + DH}{2} - \frac{DK - DH}{2} \end{array}\right\} \therefore \left\{\begin{array}{l} BD^2 = DK \cdot DH = \frac{(DK + DH)^2 - (DK - DH)^2}{4} \\ EC^2 = EH \cdot EK = \frac{(EH + EK)^2 - (EH - EK)^2}{4} \end{array}\right\} \text{..}(11)$$

$$FG = r - \sqrt{r^2 - FH^2} \text{ ......}(12)$$

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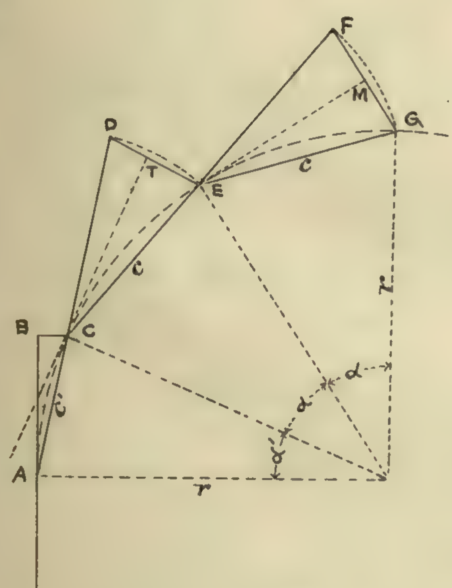
p.108
§63



$$\frac{\frac{1}{2}c'}{r} = \sin\tfrac{1}{2}\alpha' \quad \text{and} \quad \frac{\frac{1}{2}c}{r} = \sin\tfrac{1}{2}\alpha$$

$$\left.\begin{array}{l} DT = c\sin\tfrac{1}{2}\alpha' = c\,\frac{c'}{2r} \\ TE = c\sin\tfrac{1}{2}\alpha = c\,\frac{c}{2r} \end{array}\right\} \therefore$$

$$DE = DT + TE = \frac{c(c' + c)}{2r} = \frac{CE\,(AC + CD)}{2r} \quad \therefore$$

$$DE = \frac{CE \cdot AD}{2r} \text{ ....}(15)$$

$$FM = MG = c\sin\tfrac{1}{2}\alpha = c\,\frac{c}{2r} \quad \therefore$$

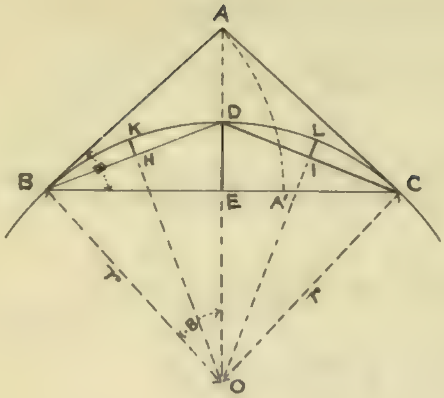
$$FG = 2FM = 2MG = 2\frac{c^2}{2r} = \frac{c^2}{r} \quad \therefore$$

$$FG = \frac{CE^2}{r} \text{ ....}(16)$$

$$BC = c'\sin\tfrac{1}{2}\alpha' = c'\,\frac{c'}{2r} = \frac{c'^2}{2r} \quad \therefore$$

$$BC = \frac{AC^2}{2r} \text{ ....}(17)$$

p 109
110
§ 63



$$ABC = BOE = B$$

$$\text{vers. in } \frac{B}{2} = 1 - \cos\frac{B}{2} = 1 - \sqrt{\frac{1+\cos B}{2}} \quad \therefore$$

$$\text{vers. in } \frac{B}{2} = 1 - \sqrt{\frac{1 + 1 - \text{vers. in } B}{2}}$$

$$\text{ver.sin } \frac{B}{2} = 1 - \sqrt{1 - \frac{\text{versin } B}{2}} \quad \text{.....}(18)$$

$$r : AB :: BE : AE \quad \therefore$$

$$r = \frac{AB \cdot BE}{AE} \quad \text{..............}(19\,A)$$

$$AE^2 = AB^2 - BE^2 = AB^2 - \frac{BC^2}{4} \quad \therefore$$

$$2\,AE = 2\sqrt{AB^2 - \frac{BC^2}{4}}$$

From (19A) $\quad r = \dfrac{AB \cdot 2BE}{2\,AE} \quad \therefore \quad r = \dfrac{AB \cdot BC}{2\sqrt{AB^2 - \frac{BC^2}{4}}}$ .... (19)

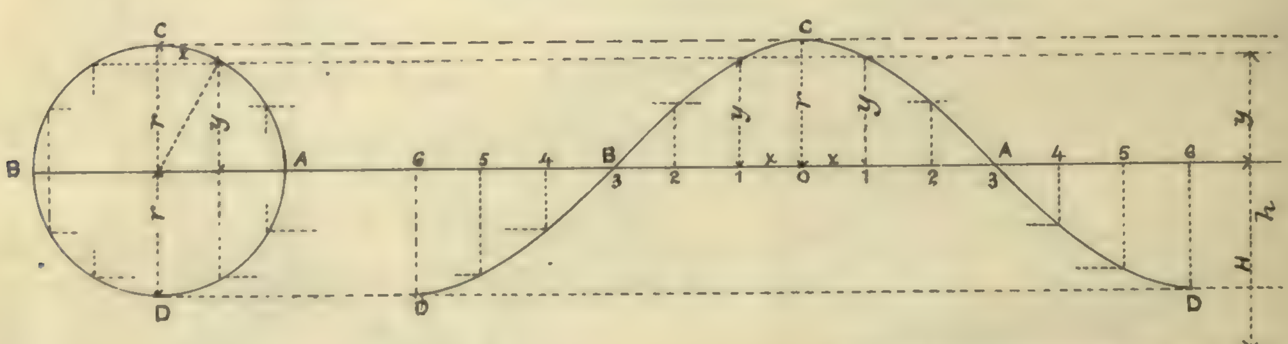
$$\text{versin } B = \frac{AB - BE}{AB} \quad \text{......................}(20)$$

$$ED = r \text{ ver.sin } B \quad \therefore \quad HK = IL = r \text{ versin } \frac{B}{2} \quad \text{..................}(21)$$

By observation it has been found that the tidal wave follows closely

p.123 the curve. $\qquad y = \cos x.$

§ 76



$$h = H + y$$

$$y = r\cos x \quad \therefore$$

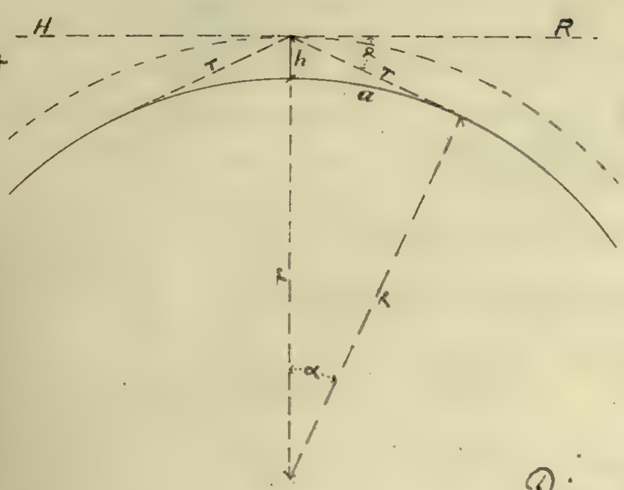
$$h = H + r\cos x$$

$$D : t :: 180 : x \quad \therefore$$

$$x^\circ = 180\,\frac{t}{D} \quad \therefore$$

$$h = H + r\cos\left(180^\circ\,\frac{t}{D}\right) \text{....................}(1)$$

p. 124
§ 80



$$n = 206264.8''$$

$$\alpha : n :: a : r \quad \therefore \quad \alpha = n\frac{a}{r}$$

$$a = T \text{ nearly} \quad \therefore \quad \alpha = n\frac{T}{r} \text{ nearly}$$

$$T^2 = (2r+h)h = 2rh + h^2 = 2rh \text{ nearly}$$

$$\therefore \quad T = \sqrt{2rh}$$

$$\alpha = n\frac{\sqrt{2rh}}{r} = n\sqrt{\frac{2hr}{r^2}} = n\sqrt{\frac{2h}{r}}$$

Refraction $= 1/10 \quad \therefore$

$$\text{Dip} = 9/10 \;\; 206264.8'' \sqrt{\frac{2h}{r}} = 57.4' \sqrt{h} \text{ in feet} \;\ldots (1)$$

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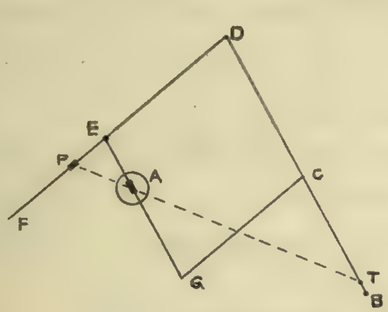
p. 126
§ 84



$$GE = ED = DC = CG$$

$$n : 1 :: TA : AP :: DE : EP \quad \therefore \;\ldots\ldots (1) \,\&\, (2)$$

$$n+1 : 1 :: DE + EP : EP :: DP : EP \quad \therefore$$

$$n+1 : 1 :: DT : EA \;\ldots\ldots\ldots (3)$$

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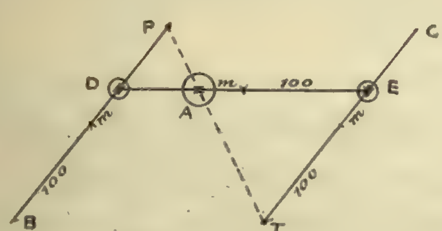
p. 127
§ 84



$$TE = EA = DB$$

$$EC = AD = DP$$

$$n : 1 :: TA : AP :: EA : AD :: \&c \;\ldots\ldots\ldots\ldots (4)$$

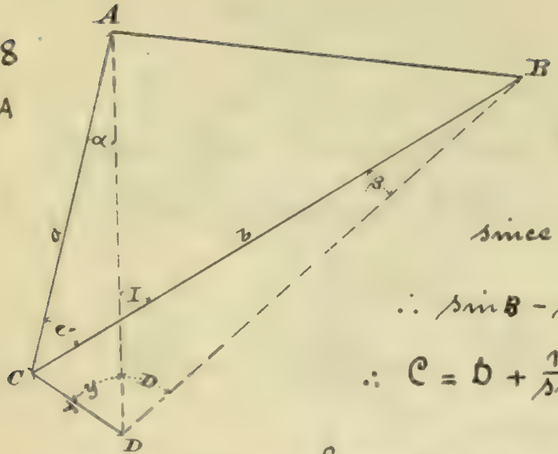
$$\left.\begin{matrix} n-1 : 1 :: EA - AD : AD \\ n+1 : 1 :: EA + AD : AD \end{matrix}\right\} \therefore \quad \frac{n-1}{n+1} = \frac{EA - AD}{EA + AD}$$

but $EA = 100 + m$ and $AD = 100 - m \quad \therefore$

$$\frac{n-1}{n+1} = \frac{(100+m)-(100-m)}{(100+m)+(100-m)} = \frac{2m}{200} \quad \therefore \quad m = 100\,\frac{n-1}{n+1} \;\ldots (5)$$

Note. The pantograph is used almost exclusively in this country and when well made its accuracy leaves nothing to be desired.
The best are made in Switzerland and cost a little over \$100, duty free.

p. 128
§86.A



$$I = \alpha + C, \quad I = \beta + D \quad \therefore \quad C - D = \beta - \alpha, \quad C = D + \beta - \alpha$$

$\sin\beta : \sin(D+y) :: r : b$ and $\sin\alpha : \sin y :: r : a$

$\therefore \sin\beta = \dfrac{r\sin(D+y)}{b}$ and $\sin\alpha = \dfrac{r\sin y}{a}$

since α and β are very small $\left\{ \begin{array}{l} \sin\alpha = \alpha \sin 1'' \\ \sin\beta = \beta\sin 1'' \end{array}\right.$

$$\therefore \sin\beta - \sin\alpha = (\beta - \alpha)\sin 1'' = \frac{r\sin(D+y)}{b} - \frac{r\sin y}{a}$$

$$\therefore C = D + \frac{r}{\sin 1''}\left\{\frac{\sin(D+y)}{b} - \frac{\sin y}{a}\right\} \qquad \text{since } \frac{1}{\sin 1''} = 206264''.8$$

we have $$ACB = ADB - 206264''.8\ CD\left\{\frac{\sin ADC}{AC} - \frac{\sin BDC}{BC}\right\} \quad \ldots\ldots (1)$$

This formula is universally applicable, A being always the first station to the right of C; or y is always measured to the right and is always less than $(D+y)$

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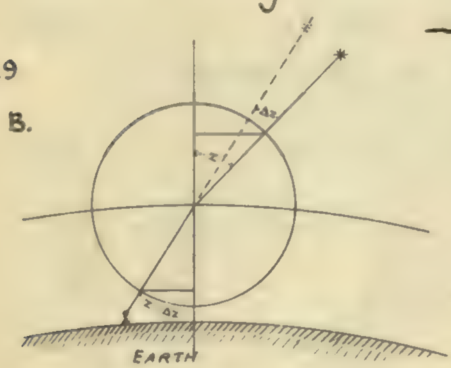
p. 129
§86 B.



Supposing the refraction to take place at one surface then by Descarte's Law

$$\frac{\sin z}{\sin(z - \Delta z)} = C, \text{ a constant}$$

$$\therefore \sin z = C(\sin z \cos\Delta z - \cos z \sin\Delta z)$$

but $\cos\Delta z = 1$ and $\sin\Delta z = \Delta z$ nearly

$$\therefore \sin z = C\sin z - C\,\Delta z\cos z$$

$$\therefore \Delta z = \frac{\sin z}{\cos z}\left(1 - \frac{1}{C}\right)$$

but $(1 - \frac{1}{C})$ has been found by observation to be $58''$ $\therefore \Delta z = r = 58'' \tan z$

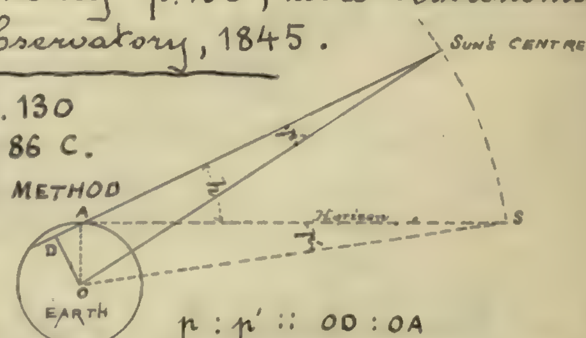
"The complete investigation of the laws of astronomical refraction is a complex problem, and one which has never been solved with entire satisfaction". It depends upon the zenith distance, the barometer, and the attached and detached thermometers.

See Doolittle's Practical Astronomy p. 153; also Astronomical Observations of the Washington Observatory, 1845.

p. 129
§86 C.

II METHOD

p. 130
§86 C.
III METHOD



$$p : p' :: OD : OA$$

$$\therefore p = p'\frac{OD}{OA}$$

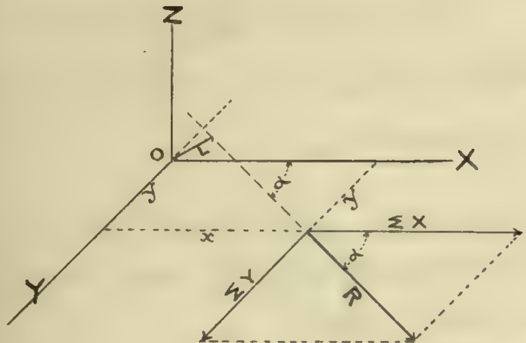
but $OD = OA \cos AOD = OA \cos h$

$$\therefore \quad p = p'\cos h$$

Rankine's Civil Engineering.
Part. II.

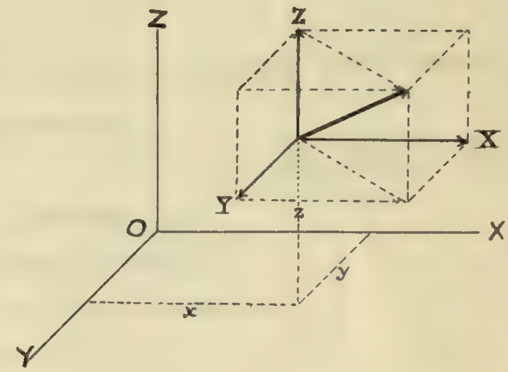
p.146

§ 99



p.147

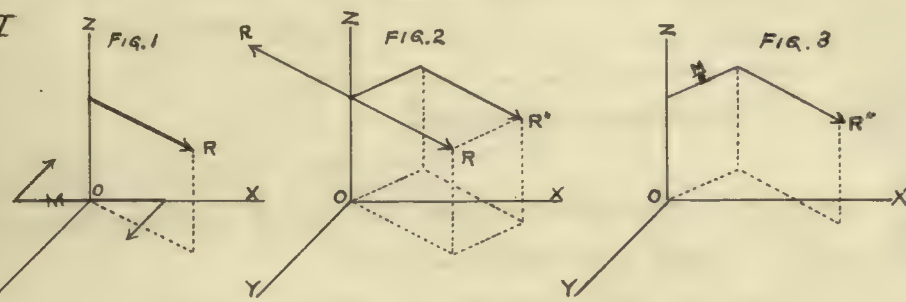
§100



CASE II

p.148

§100

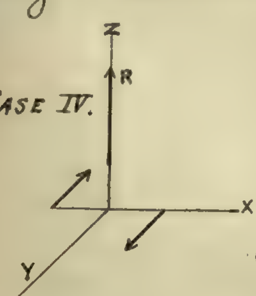


N.B. In each of these figures the plane of XY contains the moment M whose axis is Z.

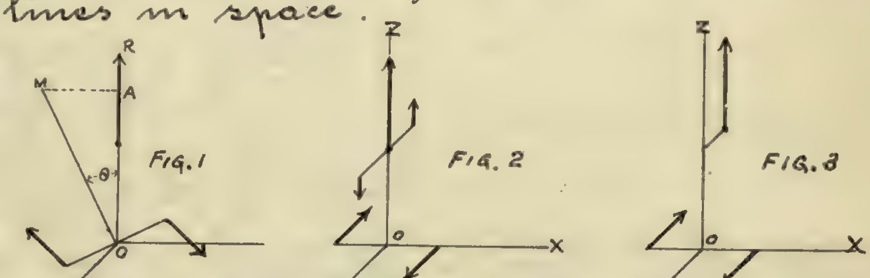
By changing the length of the lever arm make the forces of the moment each equal to R; then transfer the couple to the position shown in Fig. 2. The force R', being equal to R and in opposite direction, balances it and there remains only the force R'' = R as shown in Fig. 3

Equations (9) and (13) are the usual form for the cos. of the angle between two lines in space.

CASE IV.



CASE V.

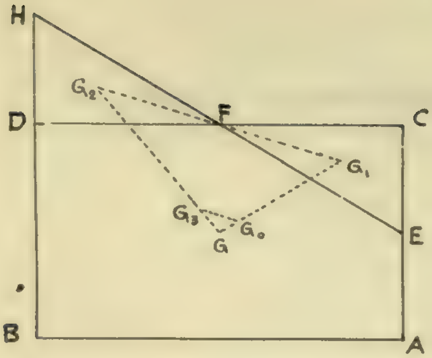


DISTANCE OM = MOMENT

MA = COMPONENT WHOSE AXIS IS PERPENDICULAR TO R

AO = " " " " PARALLEL " "

p.154 §104



$$\left.\begin{array}{l} W_0 : W_1 : W :: GG_1 : GG_0 : G_0G_1 \\ W_0 : W_1 : W :: GG_2 : GG_3 : G_2G_3 \end{array}\right\} \therefore$$

$$GG_1 : GG_0 :: GG_2 : GG_3 \quad \therefore \text{......} (A)$$

$$GG_1 : GG_0 :: G_1G_2 : G_0G_3 \quad \therefore$$

$$W_0 : W_1 :: G_1G_2 : G_0G_3$$

$$G_0G_3 = G_1G_2 \frac{W_1}{W_0} \quad \text{........} (4)$$

From the proportion (A) we see that G_0G_3 is parallel to G_1G_2.

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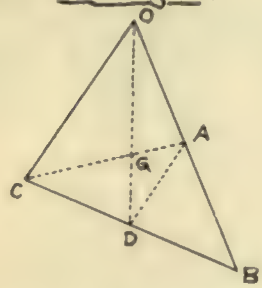
p.156 §105 I <u>Triangle</u>.



$$\left.\begin{array}{l} OA = AB \\ CD = DB \end{array}\right\}$$

$AD = \frac{1}{2}\, OC$ and $AD \parallel OC$

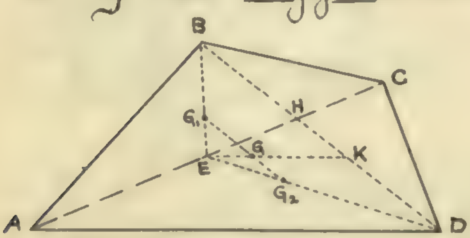
$\therefore$ The triangles OGC and AGD are similar and

$$GD = \tfrac{1}{2}\, OG \quad \therefore$$

$GD = \frac{1}{3}\, OD$ or

$$x_0 = OG = \tfrac{2}{3}\, OD .$$

II If the <u>Polygon</u> is a <u>Quadrilateral</u>.



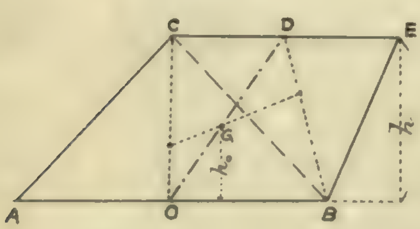
The triangles ABC and ACD are proportional to BH and HD.

$$EG_1 = \tfrac{1}{3}\, EB \;;\quad EG_2 = \tfrac{1}{3}\, ED .$$

G_1G_2 is parallel to BD; make $DK = BH$

$$ABC : ACD :: BH : HD :: DK : BK :: GG_2 : GG_1 .$$

III <u>Trapezoid</u>



h_0 = perpendicular from G to AB.

Since the triangles ACB and BCE are proportional to their bases we have

$$h_0(B+b) = \tfrac{1}{3}h\cdot B + \tfrac{2}{3}h\cdot b$$

$$\therefore\; h_0 = \frac{h}{3}\cdot\frac{B+2b}{B+b}$$

but $\quad x_0 : h_0 :: OD : h \quad \therefore\; x_0 = h_0 \dfrac{OD}{h}$

$$\therefore\; x_0 = \frac{OD}{h}\cdot\frac{h}{3}\cdot\frac{B+2b}{B+b} = \frac{OD}{3}\cdot\frac{B+2b}{B+b} \quad \therefore\; x_0 = \frac{OD}{2}\left(1 - \frac{1}{3}\cdot\frac{B-b}{B+b}\right).$$

$$W = w\cdot\frac{B+b}{2}\cdot h = w\cdot\frac{B+b}{2}\cdot OD \sin DOB = w\cdot OD\cdot\frac{B+b}{2}\cdot\sin DOB.$$

IV Trapezoid (2nd solution)

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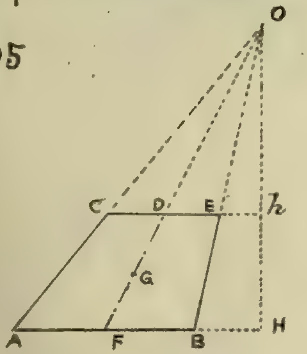
§ 105



Take the axis of moments perpendicular to the plane of the figure at O.

$OF = x_1$; $OD = x_2$; $OG = x_0$; Area $OAB = A_1$; Area $OCE = A_2$

$$A_2 : A_1 :: x_2^2 : x_1^2 \qquad \therefore \qquad A_2 = A_1 \frac{x_2^2}{x_1^2}$$

$$x_0 = \frac{\frac{2}{3}(A_1 x_1 - A_2 x_2)}{A_1 - A_2} = \frac{\frac{2}{3} A_1 \left(x_1 - \frac{x_2^3}{x_1^2}\right)}{A_1\left(1 - \frac{x_2^2}{x_1^2}\right)} = \frac{\frac{2}{3} A_1 \left(\frac{x_1^3 - x_2^3}{x_1^2}\right)}{A_1 \left(\frac{x_1^2 - x_2^2}{x_1^2}\right)} \quad \therefore$$

$$x_0 = \frac{2}{3} \frac{x_1^3 - x_2^3}{x_1^2 - x_2^2}$$

$$\text{Area } ACEB = A_1 - A_2 = A_1 \cdot \frac{x_1^2 - x_2^2}{x_1^2}$$

$$AH = OH \cot. OAB \ ; \ BH = OH \cot. OBH = - OH \cot OBA \quad \therefore$$

$$AB = AH - BH = OH (\cot OAB + \cot OBA)$$

$$A_1 = \tfrac{1}{2} AB \cdot OH = \tfrac{1}{2} OH^2 (\cot OAB + \cot OBA)$$

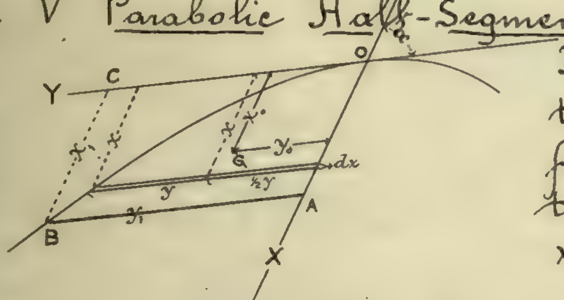
$$\text{but } OH = x_1 \sin OFB \quad \therefore$$

$$\text{Area } ACEB = \tfrac{1}{2} x_1^2 \sin^2 OFB (\cot OAB + \cot OBA) \frac{x_1^2 - x_2^2}{x_1^2} \quad \therefore$$

$$W = w \frac{x_1^2 - x_2^2}{2} \sin^2 OFB \ (\cot OAB + \cot OBA)$$

V Parabolic Half-Segment

.157
105



Here O is the origen and and AO and OC the axes. From the figure the following equations are seen to be true,

$$x_0 \sin\alpha = \frac{\int_0^{x_1} x \sin\alpha \cdot y \cdot dx \cdot \sin\alpha}{\int_0^{x_1} y \cdot dx \cdot \sin\alpha} \quad \therefore \quad x_0 = \frac{\int_0^{x_1} xy\, dx}{\int_0^{x_1} y\, dx}$$

$$y_0 \sin\alpha = \frac{\int_0^{x_1} \frac{1}{2} y \sin\alpha \cdot y \cdot dx \sin\alpha}{\int_0^{x_1} y\, dx \sin\alpha} \quad \therefore \quad y_0 = \frac{\int_0^{x_1} \frac{1}{2} y^2 dx}{\int_0^{x_1} y\, dx}$$

Since $y^2 = 2px$

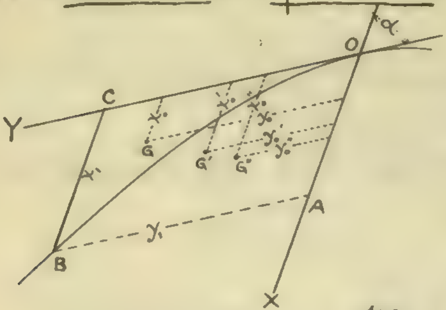
$$x_0 = \frac{\int_0^{x_1} \sqrt{2p}\, x^{\frac{3}{2}}\, dx}{\int_0^{x_1} \sqrt{2p}\, x^{\frac{1}{2}}\, dx} = \frac{\frac{2}{5} x_1^{\frac{5}{2}}}{\frac{2}{3} x_1^{3/2}} = \frac{3}{5} x_1$$

$$y_0 = \frac{\int_0^{x_1} 2px\, dx}{2\int_0^{x_1} \sqrt{2p}\, x^{\frac{1}{2}} dx} = \frac{\sqrt{2p}\, \frac{x_1^2}{2}}{2 \cdot \frac{2}{3} x_1^{3/2}} = \frac{3}{8} \sqrt{2px_1} = \frac{3}{8} y_1$$

$$W = w \int y\, dx \sin\alpha = w \sin\alpha \sqrt{2p} \int x^{1/2} dx = w \sin\alpha \sqrt{2p}\, \tfrac{2}{3} x_1^{3/2} \ ; \ \text{but } \sqrt{2p} = \frac{y_1}{\sqrt{x}} \quad \therefore$$

$$W = \tfrac{2}{3} w\, x_1 y_1 \sin\alpha$$

p.157 §104 VI <u>Parabolic Spandril.</u>



$W' =$ weight of parallelogram $OABC = w\, x_1 y_1 \sin\alpha$

$W'' = \frac{2}{3} W' =$ weight of Segment $OAB = \frac{2}{3} w\, x_1 y_1 \sin\alpha$

$W = \frac{1}{3} W' =$ " " Spandril $OBC = \frac{1}{3} w\, x_1 y_1 \sin\alpha$

$$\left.\begin{aligned} x_0' &= \tfrac{1}{2} x_1 \\ x_0'' &= \tfrac{3}{5} x_1 \end{aligned}\right\} \quad \left\{\begin{aligned} y_0' &= \tfrac{1}{2} y_1 \\ y_0'' &= \tfrac{3}{8} y_1 \end{aligned}\right.$$

$$W x_0 = W' x_0' - W'' x_0''$$
$$\tfrac{1}{3} W' x_0 = W' \cdot \tfrac{1}{2} x_1 - \tfrac{2}{3} W' \cdot \tfrac{3}{5} x_1$$
$$\therefore x_0 = \frac{3}{2} x_1 - \frac{6}{5} x_1 = \frac{3}{10} x_1$$

$$W y_0 = W' y_0' - W'' y_0''$$
$$\tfrac{1}{3} W' y_0 = W' \tfrac{1}{2} y_1 - \tfrac{2}{3} W' \cdot \tfrac{3}{8} y_1$$
$$\therefore y_0 = \frac{3}{2} y_1 - 2 \cdot \frac{3}{8} y_1 = \frac{3}{4} y_1$$

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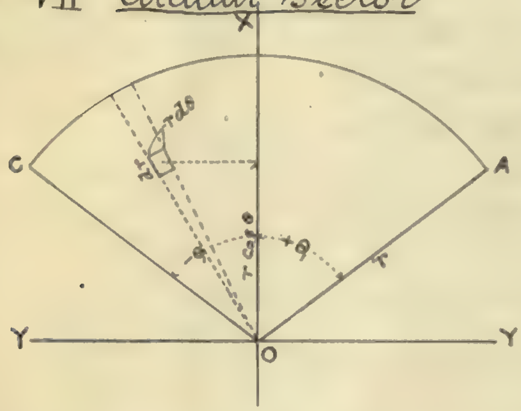
p.157 §104 VII <u>Circular Sector</u>



Elementary area $= r\, dr \cdot d\theta$

" moment $= r^2 dr \cdot \cos\theta\, d\theta$

$$x_0 = \frac{\int_0^r r^2 dr \int_{-\theta}^{+\theta} \cos\theta\, d\theta}{\int_0^r r\, dr \int_{-\theta}^{+\theta} d\theta} \quad \therefore$$

$$x_0 = \frac{\frac{r^3}{3} \cdot 2 \sin\theta}{\frac{r^2}{2} \cdot 2\theta} = \frac{2}{3} r \frac{\sin\theta}{\theta}$$

$$W = w\, r^2 \theta$$

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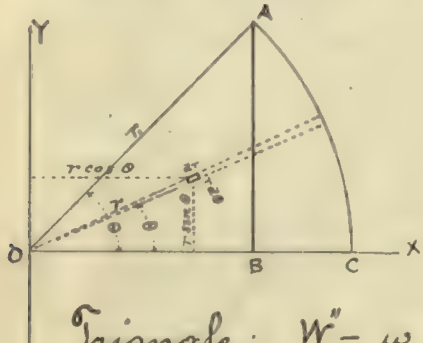
p 158 §104 <u>Circular Half-Segment</u>



<u>Sector.</u> $W' = w\, OAC = w\, \frac{1}{2} r^2 \theta$

$$x_0' = \frac{\int_0^r r^2 dr \int_0^\theta \cos\theta\, d\theta}{\int_0^r r\, dr \int_0^\theta d\theta} = \frac{2}{3} r \frac{\sin\theta}{\theta}$$

$$y_0' = \frac{\int_0^r r^2 dr \int_0^\theta \sin\theta\, d\theta}{\int_0^r r\, dr \int_0^\theta d\theta} = \frac{2}{3} r \frac{1 - \cos\theta}{\theta} = \frac{2}{3} r \frac{2 \sin^2 \frac{\theta}{2}}{\theta}$$

<u>Triangle</u>; $W'' = w\, \frac{1}{2} r^2 \sin\theta \cos\theta$; $x_0'' = \frac{2}{3} r \cos\theta$; $y_0'' = \frac{1}{3} r \sin\theta$

$$\therefore x_0 = \frac{W' x_0' - W'' x_0''}{W' - W''} = \frac{\frac{1}{2} r^2 \theta \cdot \frac{2}{3} r \frac{\sin\theta}{\theta} - \frac{1}{2} r^2 \sin\theta \cos\theta \cdot \frac{2}{3} r \cos\theta}{\frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin\theta \cos\theta} = \frac{2}{3} r \frac{\sin^3\theta}{\theta - \sin\theta\cos\theta}$$

$$y_0 = \frac{W' y_0' - W'' y_0''}{W' - W''} = \frac{\frac{1}{2} r^2 \theta \cdot \frac{2}{3} r \frac{2 \sin^2 \frac{\theta}{2}}{\theta} - \frac{1}{2} r^2 \sin\theta \cos\theta \cdot \frac{1}{3} r \sin\theta}{\frac{1}{2} r^2 (\theta - \sin\theta \cos\theta)} =$$

$$y_0 = \frac{1}{3} r \frac{4 \sin^2 \frac{\theta}{2} - \sin^2\theta \cos\theta}{\theta - \sin\theta \cos\theta}$$

IX Circular Spandril

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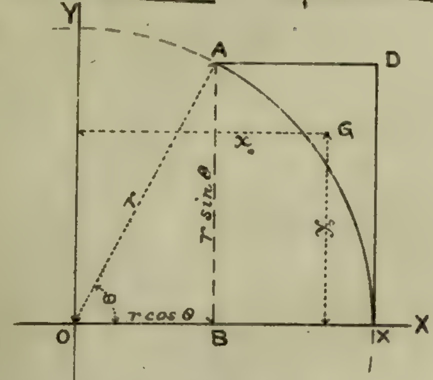


$$ADX = ABXD - ABX =$$

$$r\sin\theta(1-\cos\theta)r - \frac{r^2(\theta - \sin\theta\cos\theta)}{2} =$$

$$r^2\left(\sin\theta - \frac{\sin\theta\cos\theta}{2} - \frac{\theta}{2}\right) \therefore$$

$$W = w r^2\left(\sin\theta - \frac{\sin\theta\cos\theta}{2} - \frac{\theta}{2}\right)$$

$$x_o r^2\left(\sin\theta - \frac{\sin\theta\cos\theta}{2} - \frac{\theta}{2}\right) = \left\{ r\sin\theta(1-\cos\theta)r\left\{r\cos\theta + \frac{r(1-\cos\theta)}{2}\right\} - \frac{r^2(\theta-\sin\theta\cos\theta)}{2}\cdot\frac{2}{3}r\frac{\sin^3\theta}{\theta - \sin\theta\cos\theta} \right\} =$$

$$= r^2(\sin\theta - \sin\theta\cos\theta)\left(\cos\theta + \frac{1-\cos\theta}{2}\right)r - \frac{r^3}{3}\sin^3\theta = \frac{r^3}{2}\sin^3\theta - \frac{r^3}{3}\sin^3\theta \quad \therefore$$

$$x_o r^2\left(\sin\theta - \frac{\sin\theta\cos\theta}{2} - \frac{\theta}{2}\right) = \frac{r^3}{6}\sin^3\theta$$

$$x_o = \frac{r}{3}\,\frac{\sin^3\theta}{2\sin\theta - \sin\theta\cos\theta - \theta}.$$

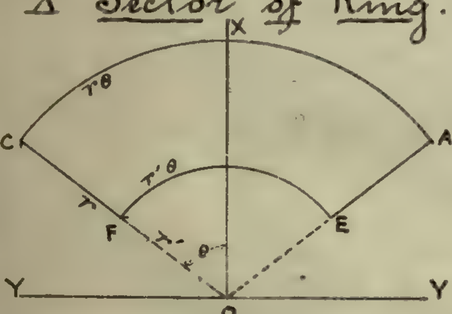
$$y_o r^2\left(\sin\theta - \frac{\sin\theta\cos\theta}{2} - \frac{\theta}{2}\right) = r\sin\theta(1-\cos\theta)r\,\frac{r\sin\theta}{2} - \frac{r^2(\theta - \sin\theta\cos\theta)}{2}\cdot\frac{r}{3}\,\frac{4\sin^2\frac{\theta}{2} - \sin^2\theta\cos\theta}{\theta - \sin\theta\cos\theta}$$

$$= \frac{r^3}{2}\sin^2\theta(1-\cos\theta) - \frac{r^3}{6}\left(4\sin^2\frac{\theta}{2} - \sin^2\theta\cos\theta\right)$$

$$= \frac{r^3}{6}\left(3\sin^2\theta - 2\sin^2\theta\cos\theta - 4\sin^2\frac{\theta}{2}\right)$$

$$\therefore \quad y_o = \frac{r}{3}\,\frac{3\sin^2\theta - 2\sin^2\theta\cos\theta - 4\sin^2\frac{\theta}{2}}{2\sin\theta - \sin\theta\cos\theta - \theta}$$

X Sector of Ring.



$$ACFE = OAC - OEF.$$

$$\text{area } OAC = \frac{2r\theta\cdot r}{2}$$

$$\text{〃} \quad OEF = \frac{2r'\theta\cdot r'}{2}$$

$$\therefore ACFE = r^2\theta - r'^2\theta = \theta(r^2 - r'^2)$$

x''_o = abscissa of C of G. of $OAC = \frac{2}{3}r\frac{\sin\theta}{\theta}$

x'_o = 〃 〃 〃 〃 $OEF = \frac{2}{3}r'\frac{\sin\theta}{\theta}$

$$x_o\cdot\theta(r^2 - r'^2) = r^2\theta\cdot\frac{2}{3}r\frac{\sin\theta}{\theta} - r'^2\theta\cdot\frac{2}{3}r'\frac{\sin\theta}{\theta}$$

$$x_o\cdot\theta(r^2 - r'^2) = \frac{2}{3}r^3\sin\theta - \frac{2}{3}r'^3\sin\theta$$

$$\therefore \quad x_o = \frac{2}{3}\,\frac{r^3 - r'^3}{r^2 - r'^2}\cdot\frac{\sin\theta}{\theta}$$

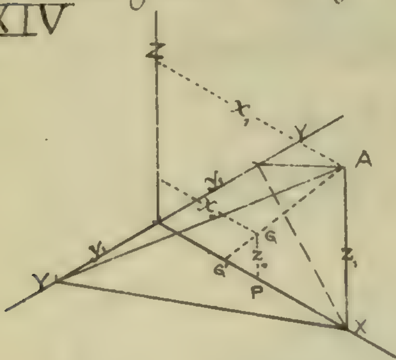
$$y_o = 0$$

$$W = w(r^2 - r'^2)\theta.$$

Triangular Wedge.

XIV

p. 159
§ 105



$$y : y_1 :: x_1 - x : x_1 \quad \therefore \quad y = \frac{y_1}{x_1}(x_1 - x)$$

$$x_0 \int_0^{x_1} xy\,dx = \int_0^{x_1} x^2 y\,dx$$

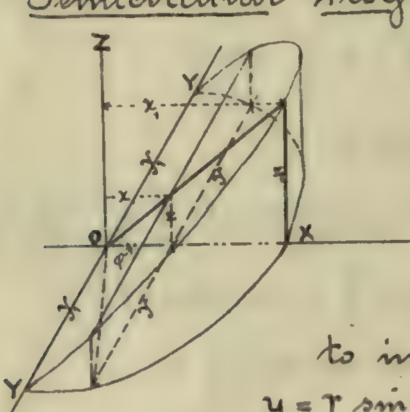
$$x_0 \frac{y_1}{x_1}\int_0^{x_1} x(x_1 - x)\,dx = \frac{y_1}{x_1}\int_0^{x_1} x^2(x_1 - x)\,dx$$

$$x_0\left[\tfrac{1}{2}x_1 x_1^2 - \tfrac{1}{3}x_1^3\right] = \tfrac{1}{3}x_1 x_1^3 - \tfrac{1}{4}x_1^4$$

$$\therefore \quad x_0 = \tfrac{1}{2}x_1$$

$$z = \frac{z_1 x_0}{2\,x_1} = \tfrac{1}{4}z_1$$

XV Semicircular Wedge



$$x^2 + y^2 = r^2 \quad \therefore x\,dx = -y\,dy$$

From XII we have, $x_0 = \dfrac{\int x^2 y\,dx}{\int xy\,dx}$

$$\therefore x_0 \int xy\,dx = \int x^2 y\,dx$$

$$-x_0 \int y^2 dy = \int x^2 y\,dx \,;$$

to integrate the second member let

$$y = r\sin\varphi\,,\quad x = r\cos\varphi\,,\quad dx = -r\sin\varphi\,d\varphi$$

$$\therefore -x_0 \int_{-r}^{+r} y^2 dy = -\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} r^4 \sin^2\varphi\cos^2\varphi\,d\varphi$$

$$x_0 \cdot \tfrac{2}{3}r^3 = r^4 \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin^2\varphi\cos^2\varphi\,d\varphi$$

[see text p. 40 Eq 11],

but $\sin^2\varphi = \dfrac{1-\cos 2\varphi}{2}$ and $\cos^2\varphi = \dfrac{1+\cos 2\varphi}{2}$ $\therefore$

$$\sin^2\varphi\cos^2\varphi = \frac{1-\cos^2 2\varphi}{4} = \frac{\sin^2 2\varphi}{4} = \frac{1-\cos 4\varphi}{8}$$

$$\therefore x_0 \cdot \tfrac{2}{3}r^3 = \frac{r^4}{8}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\left[d\varphi - \tfrac{1}{4}\cos 4\varphi \cdot 4\,d\varphi\right]$$

$$= \frac{r^4}{8}\left[\frac{\pi}{2} + \frac{\pi}{2} - \tfrac{1}{4}\sin 4\frac{\pi}{2} + \tfrac{1}{4}\sin 4\left(-\frac{\pi}{2}\right)\right]$$

$$\therefore \quad x_0 = \tfrac{3}{16}\pi r \,;\quad y_0 = 0 \,;\quad z_0 = \frac{z_1}{2x_1}\cdot\tfrac{3}{16}\pi r = \frac{z_1}{2r}\cdot\tfrac{3}{16}\pi r = \tfrac{3}{32}\pi z_1 .$$

$$W = w\int zy\,dx = w\frac{z_1}{x_1}\int xy\,dx \,,$$

$$= -w\frac{z_1}{r}\int_{-r}^{+r} y^2 dy = -w\frac{z_1}{r}\left(\tfrac{1}{3}r^3 + \tfrac{1}{3}r^3\right) = -\tfrac{2}{3}w z_1 r^2 .$$

XVI Annular, or Hollow Semicircular Wedge.

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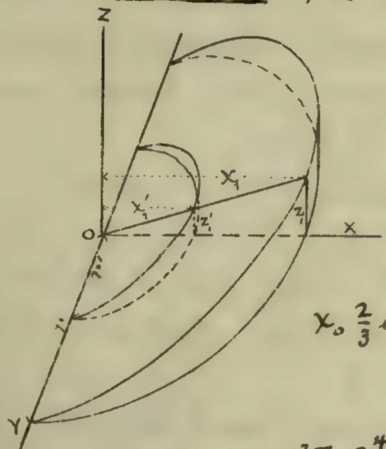

$W' =$ weight of large wedge.

$W'' =$ " " small "

$W =$ " " annular "

$x_0(W' - W'') = W'x_0' - W''x_0''$; and since $z_1' = \frac{z_1}{r} r'$ we have

$$x_0 \tfrac{2}{3} w\left(r^2 z_1 - r'^2 \tfrac{z_1}{r} r'\right) = \tfrac{2}{3} w\left(r^2 z_1 \cdot \tfrac{3}{16}\pi r - r'^2 \tfrac{z_1}{r} r' \cdot \tfrac{3}{16}\pi r'\right)$$

$$x_0 \frac{r^3 - r'^3}{r} = \frac{3\pi}{16}\cdot\frac{r^4 - r'^4}{r} \quad \therefore$$

$$x_0 = \frac{3\pi}{16}\cdot\frac{r^4 - r'^4}{r^3 - r'^3}\ ; \quad y_0 = 0\ ; \quad z_0 = \frac{3\pi}{32}\cdot\frac{z'}{r}\cdot\frac{r^4 - r'^4}{r^3 - r'^3}$$

$$W = \frac{2}{3} w\left(\frac{r^3 z_1 - r'^3 z_1}{r}\right) = \frac{2}{3} w z_1 \frac{r^3 - r'^3}{r}$$

XVII Complete Cone or Pyramid.

1.160

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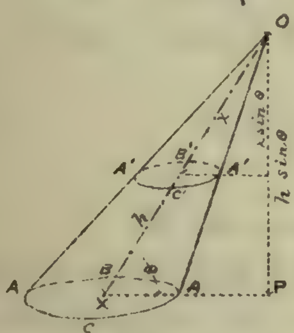

Take the axis of moments through O perpendicular to the plane OXP.

Area ABAC = A = area of base.

For the area of any section parallel to the base we have

$$A' : A :: x^2 : h^2 \quad \therefore A' = A\frac{x^2}{h^2}$$

Elementary volume $= A\frac{x^2}{h^2}\,dx \sin\theta$

" moment $= A\frac{x^2}{h^2}\,dx \sin\theta \cdot x$

$$\therefore x_0 = \frac{\frac{A}{h^2}\sin\theta \cdot \int_0^h x^3 dx}{\frac{A}{h^2}\sin\theta \cdot \int_0^h x^2 dx} = \frac{\frac{1}{4}h^4}{\frac{1}{3}h^3} = \frac{3}{4}h$$

$$W = w\frac{A}{h^2}\sin\theta \int_0^h x^2 dx = w\frac{A}{h^2}\sin\theta \cdot \frac{h^3}{3} = \frac{1}{3} w A h \sin\theta.$$

XVIII Truncated Cone or Pyramid.

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SEE FIGURE ABOVE

Taking the above integral between the limits h' and h we have

$$x_0 = \frac{\frac{A}{h^2}\sin\theta \cdot \int_{h'}^{h} x^3 dx}{\frac{A}{h^2}\sin\theta \cdot \int_{h'}^{h} x^2 dx} = \frac{\frac{1}{4}(h^4 - h'^4)}{\frac{1}{3}(h^3 - h'^3)} = \frac{3}{4}\cdot\frac{h^4 - h'^4}{h^3 - h'^3}.$$

$$W = w\frac{A}{h^2}\sin\theta \int_{h'}^{h} x^2 dx = \frac{1}{3} w A h \cdot \left(1 - \frac{h'^3}{h^3}\right)\sin\theta.$$

Zone or Ring of a Spherical Shell.

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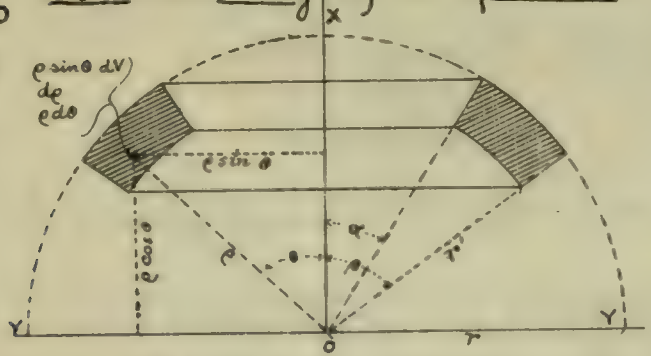

SECTION BY PLANE XY

N.B. THE AXIS Z IS PERP. TO PLANE OF THE PAPER

V is estimated from $0°$ to $360°$ in a plane perpendicular to OX, with radius $= \rho \sin\theta$.
Arm of moments $= r\cos\theta$.

$$\text{El. Vol.} = \rho^2 d\rho \cdot \sin\theta\, d\theta \cdot dV$$
$$\text{El. momt.} = \rho^3 d\rho \cdot \sin\theta\cos\theta\, d\theta \cdot dV$$

$$x_0 = \frac{\int_{r'}^{r} \rho^3 d\rho \int_\alpha^\beta \sin\theta\cos\theta\, d\theta \int_0^{2\pi} dV}{\int_{r'}^{r} \rho^2 d\rho \int_\alpha^\beta \sin\theta\, d\theta \int_0^{2\pi} dV}$$

$$x_0 = \frac{\frac{1}{4}(r^4 - r'^4)\cdot\frac{1}{2}\left(\sin^2\beta - \sin^2\alpha\right) 2\pi}{\frac{1}{3}(r^3 - r'^3)\cdot(-\cos\beta + \cos\alpha)\, 2\pi} = \frac{3}{4}\cdot\frac{r^4 - r'^4}{r^3 - r'^3}\cdot\frac{\cos^2\beta - \cos^2\alpha}{2(\cos\beta - \cos\alpha)}$$

$$= \frac{3}{4}\cdot\frac{r^4 - r'^4}{r^3 - r'^3}\cdot\frac{\cos\beta + \cos\alpha}{2}.$$

$$W = w\int_r^{r'} \rho^2 d\rho \int_\alpha^\beta \sin\theta\, d\theta \int_0^{2\pi} dV = \frac{2\pi w}{3}\cdot(r^3 - r'^3)\,(-\cos\beta + \cos\alpha)$$

XX Sector of a Hemispherical Shell.

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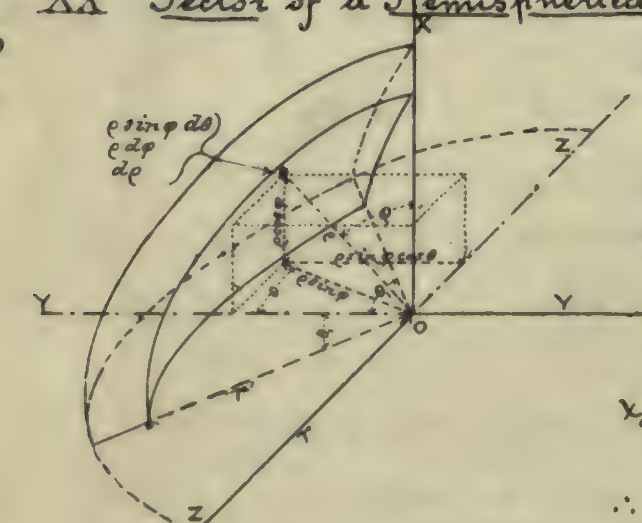
§105



$$\text{El. Vol.} = \rho^2 d\rho \cdot \sin\varphi\, d\varphi \cdot d\theta$$
$$\text{El. M(axis in ZOY)} = \rho^3 d\rho \cdot \sin\varphi\cos\varphi\, d\varphi \cdot d\theta$$
$$\text{,, ,, (,, XOZ)} = \rho^3 d\rho \cdot \sin^2\varphi\, d\varphi \cdot \cos\theta\, d\theta$$

$$x_0 = \frac{\int_{r'}^{r} \rho^3 d\rho \int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi\, d\varphi \int_{-\theta_1}^{+\theta_1} d\theta}{\int_{r'}^{r} \rho^2 d\rho \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \int_{-\theta}^{+\theta} d\theta}$$

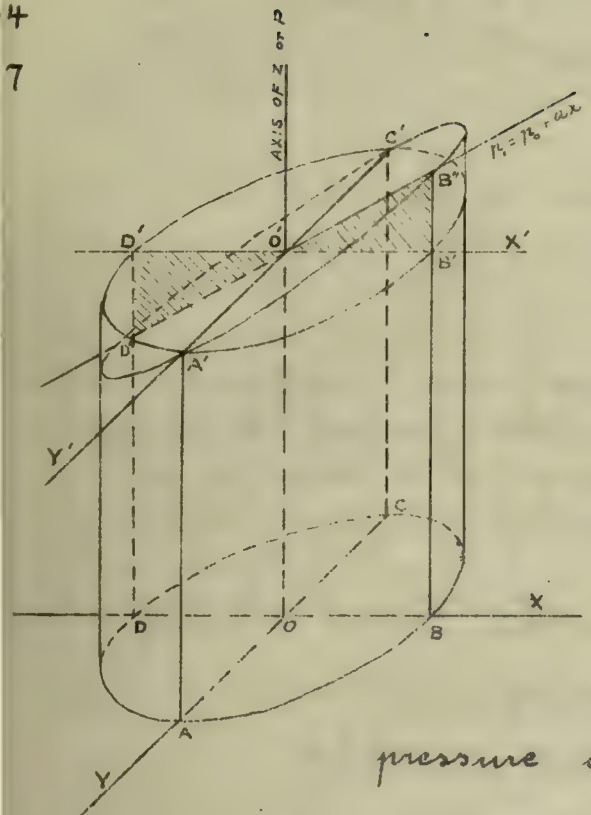
$$x_0 = \frac{\frac{1}{4}(r^4 - r'^4)\,\frac{1}{2}\left(\sin^2\frac{\pi}{2} - \sin^2 0\right)(\theta_1 + \theta_1)}{\frac{1}{3}(r^3 - r'^3)\left(-\cos\frac{\pi}{2} + \cos 0\right)(\theta_1 + \theta_1)}$$

$$\therefore\ x_0 = \frac{3}{8}\cdot\frac{r^4 - r'^4}{r^3 - r'^3}.$$

$$y_0 = \frac{\int_{r'}^{r} \rho^3 d\rho \int_0^{\frac{\pi}{2}} \sin^2\varphi\, d\varphi \int_{-\theta_1}^{+\theta_1} \cos\theta\, d\theta}{\int_{r'}^{r} \rho^2 d\rho \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \int_{-\theta_1}^{+\theta_1} d\theta}\ ; \quad \text{but } \int_0^{\frac{\pi}{2}} \sin^2\varphi\, d\varphi = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1 - \frac{2\cos 2\varphi}{2}\right) d\varphi = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - (\sin\pi \ldots$$
$$= \frac{1}{4}\pi$$

$$\therefore\ y_0 = \frac{\frac{1}{4}(r^4 - r'^4)\cdot\frac{1}{4}\pi\cdot(\sin\theta_1 + \sin\theta_1)}{\frac{1}{3}(r^3 - r'^3)\left(-\cos\frac{\pi}{2} + \cos 0\right)(\theta_1 + \theta_1)} = \frac{3\pi}{16}\cdot\frac{r^4 - r'^4}{r^3 - r'^3}\cdot\frac{\sin\theta_1}{\theta_1}.$$

$$W = \frac{2\theta_1}{3}\, w(r^3 - r'^3)$$



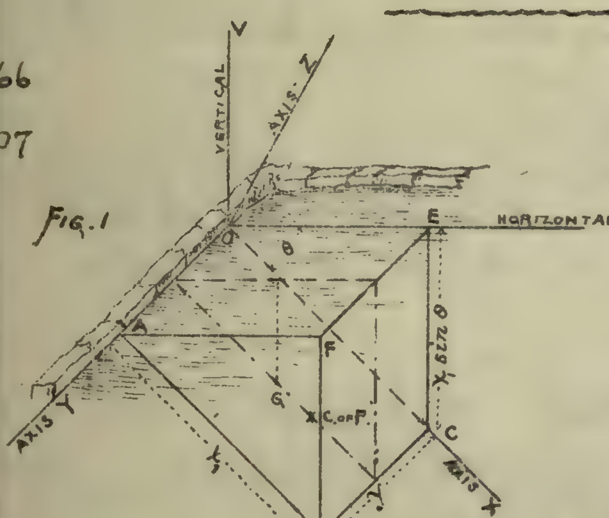
If the pressure on ABCD is uniform it is represented by the cylinder with parallel bases ABCD-A′B′C′D′ where evidently p = a constant.

If the pressure is uniformly varied, as is the case with fluid pressure, it is represented by the cylinder ABCD-A′B″CD″; the pressure in planes parallel to YZ being constant and in planes parallel to ZX the pressure varies as shown by the shaded portion or

$$p_1 = p_0 + ax$$

where p_0 may be either the mean pressure as is evident from the figure or it may be an initial pressure as in Eq.(1) of the text.

FIG. 1



From the figure it may be seen that the mean pressure in lbs. pr. sq. ft is = w × the mean ordinate of the triangle OCE = $\frac{w x_1 \sin\theta}{2}$

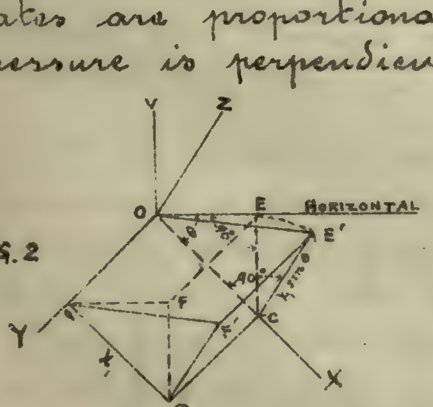
$$\therefore \frac{P}{S} = \frac{w x_1 \sin\theta}{2}$$

and since $w = 62.4$ $\quad \frac{P}{S} = \frac{62.4\, x_1 \sin\theta}{2}$ (8)

$$S = x_1 y \quad \therefore \; P = \frac{62.4\, x_1^2 y \sin\theta}{2} \quad \ldots\ldots (9)$$

In the figure it must be remembered that while the vertical ordinates are proportional to the pressure the direction of the pressure is perpendicular to the plane of XY, or ABCO.

FIG. 2



To represent the direction as well as the intensity CE′ must be drawn parallel to Z and equal CE = $x_1 \sin\alpha = \frac{p}{w}$.

The centre of pressure is found by finding the centre of gravity of the triangle OCE or OCE′ and projecting it on the plane YX which gives $x_0 = \frac{2}{3} x_1$.

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Cont.

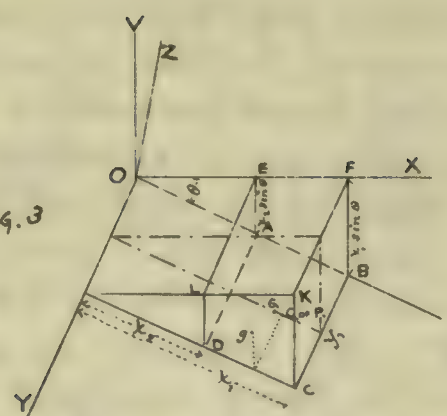


Fig. 3

Or the centre of pressure may be found directly, $x_0 = \dfrac{\int_0^{x_1} x^2 y\,dx}{\int_0^{x} xy\,dx}$, since y in this case is constant, we have

$$x_0 = \frac{\frac{1}{3}x_1^3 y}{\frac{1}{2}x^2 y} = \frac{2}{3}x_1 \text{ ......} \quad (10)$$

If the upper edge is at a distance x_2 from the surface, as in fig. 3, the distance of G from OY is

$$x_2 + \frac{x_1 - x_2}{2} = \frac{x_1 + x_2}{2}$$

The mean pressure pr. sq. ft. is $\dfrac{P}{S} = \dfrac{w(x_1+x_2)\sin\theta}{2} = \dfrac{62.4(x_1+x_2)\sin\theta}{2}$

this is the mean ordinate of the trapezoid ABFE $\times w$

$$S = y(x_1 - x_2) \quad \therefore P = \frac{62.4(x_1^2 - x_2^2)\sin\theta\, y}{2}$$

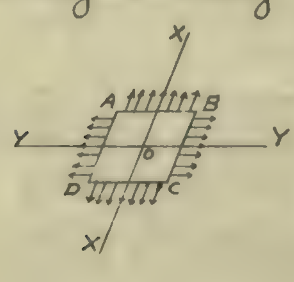
By subtraction, or by integrating the above between the limits x_2 and x_1, we have

$$x_0 = \frac{2}{3}\,\frac{x_1^3 - x_2^3}{x_1^2 - x_2^2}.$$

This may be found graphically by finding the c. of G. of the trapezoid DCKL and projecting it as shown in the figure.

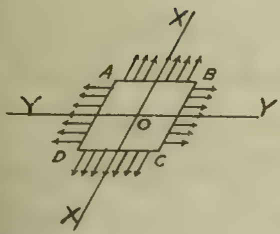
Internal Stress of Solids.

p. 166
§108 For a full discussion of the subject the student is referred to the App. Mech. and to a small book by Prof. Alexander [N.Y. McMillan & Co.]. Only so much is given in these notes as is necessary to enable the student to understand the equations used by our author in the Civ. Engineering.



Pair of Conjugate Stresses. [A. M. 101]

"Let YOY represent, in section a given plane traversing a body, and let the stress on that plane be in the direction of XOX. Consider the condition of a prismatic portion of the body represented in section

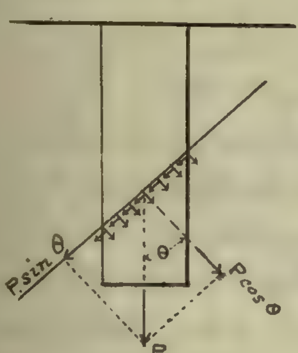


by ABCD, bounded by a pair of planes AB, DC, parallel to the given plane, and a pair of planes AD, BC, parallel to each other and to the given direction XOX and having for its axis a line in the plane YOY, cutting XOX in O."

"The equal resultant forces exerted by the other parts of the body on the faces AB and DC of this prism are directly opposed, their common line of action traversing the axis O; and they are therefore independently balanced. Therefore the forces exerted by the other parts of the body on the faces AD and BC of the prism must be independently balanced and have their results directly opposed; which cannot be unless their direction is parallel to the plane YOY."

II TANGENTIAL AND NORMAL STRESS.

(See *Applied Mechanics*, by Jas. H. Cotterill § 182)



"Let us take the case of a stretched bar carrying a load P. On a transverse section only a normal stress is produced. Now suppose we take an oblique section, whose normal makes an angle θ with the axis of the bar, and let us resolve the force P into two components, one perpendicular and the other parallel to the section. The normal component $P\cos\theta$ tends to produce a direct separation at the section, producing a tensile stress similar in character to that on a transverse section but of less intensity.

If A = Area of transverse section of bar, $A\sec\theta$ = Area of Oblique section; the intensity of the normal stress

$$p_n = \frac{P\cos\theta}{A\sec\theta} = \frac{P}{A}\cos^2\theta = p\cos^2\theta\text{, where } p = \frac{P}{A}.$$

The other component $P\sin\theta$ produces a tangential or shearing stress of intensity

$$p_t = \frac{P\sin\theta}{A\sec\theta} = p\sin\theta\cos\theta.$$

Similarly if a bar is subjected to a compressive instead

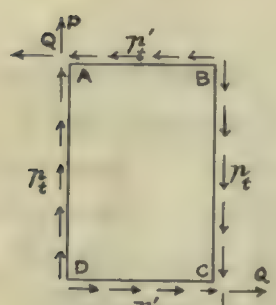
p. 167 of a tensile load."

§.108 "Any material which offers great resistance to direct compression yields by sliding across an oblique plane.

Conx. Now p_t is a maximum when $\theta = 45°$ this is therefore approximately the angle of separation. The same maximum stress, the value of which is $\frac{p}{2}$, occurs on another plane sloping the other way at an angle of 45°. We sometimes find fracture to occur across two oblique planes; sometimes across one only."

"If in $p_t = p \sin\theta \cos\theta$ we change θ into $90+\theta$, p_t has the same value; so that the intensity of the tangential stresses on two planes at right angles to each other is the same. This is general in all cases of stress as will be seen presently."

TANGENTIAL STRESS EQUIVALENT TO A PAIR OF EQUAL AND OPPOSITE NORMAL STRESSES. <u>Distorting</u> <u>Stress</u>.



In the example we have just considered we have both shearing and normal stress; but there are some cases in which there is only a shearing stress. Let ABCD be a rectangular plate of thickness t. Over the surfaces BC and AD suppose a tangential stress to be applied of intensity p_t. Calling b and a the length of the sides of the plate, the total amount of the tangential stress on each side is

$$P = p_t \cdot bt.$$

To prevent the turning of the plate, suppose the forces P balanced by the application of an uniform stress over the surfaces AB and DC of intensity p_t'. The amount of the force on each of these sides.

$$Q = p_t' \cdot a \cdot t.$$

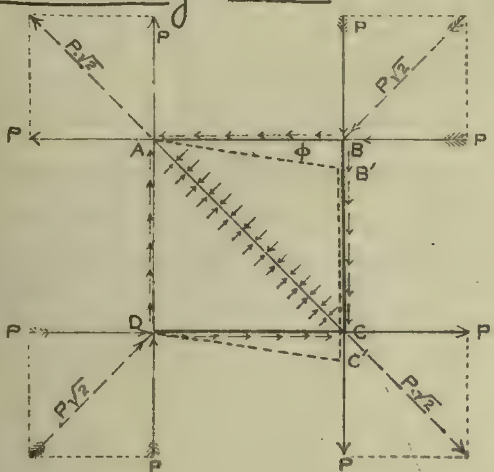
Since equilibrium is produced the moment of the couple P must be equal to the couple Q.

$$\therefore p_t \cdot bt \cdot a = p_t' \cdot at \cdot b$$

$$\text{or} \quad p_t = p_t' ;$$

That is the intensity of the stress is the same on AB as on AD.

167 Shearing therefore cannot exist along one plane only
108 but must be accompanied by a shearing stress of equal
intensity in a plane at right angles. Such a pair of stres-
ont. ses, unaccompanied by normal stress, constitute a <u>Simple Distorting Stress</u>.



Let us now assume for simplicity the plate to be square. The effect of the forces is to produce a change of form which, in perfectly elastic bodies, is exactly proportional to the shearing force which produces it. The square ABCD becomes the rhombus AB'C'D, the angle of distortion φ being proportional to the stress p_t. We may write

$$p_t = C\varphi,$$

where the coëfficient C is a kind of Modulus of Elasticity, but of a different kind from the ordinary (Young's) Modulus of Elasticity, E, which we will subsequently consider.

C is sometimes called modulus of transverse elasticity but preferably the <u>Coëfficient</u> of <u>Rigidity</u>.

C for metallic bodies is generally less than $\frac{2}{5}$ E, and for wrought-iron bars may be taken as 10 to 10½ millions.

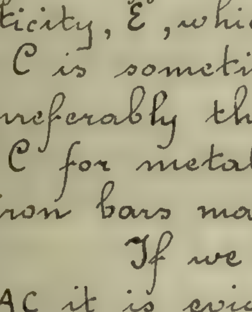
If we now consider the stress on the diagonal section AC it is evident there will be no tangential component but that the two surfaces will be pressed together by a force

$$\sqrt{P^2 + P^2} = P\sqrt{2}.$$

If we divide this force by the area over which it is distributed we get the intensity

$$\frac{P\sqrt{2}}{at\sqrt{2}} = \frac{P}{at} = p_t.$$

This force is compressive on the diagonal AC and tensile on BD the intensity being the same for each and $= p_t$



Thus it appears that a shearing stress necessarily involves tensile and compressive stresses on planes at right angles to each other and inclined at an angle of 45° to the plane of the shearing stress.

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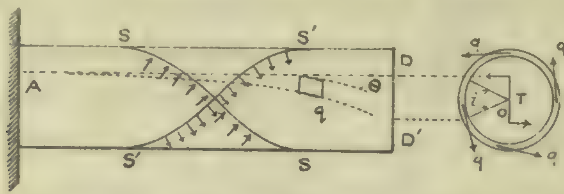
TORSION [Cotterill's App. Mec. p. 358, §183].

Torsion is produced by a pair of equal and opposite couples applied to the ends of a bar, the axis of the couples coinciding with the axis of the bar.

When we consider the elastic force called into action torsion reduces to a case of shearing.



Taking a simple case imagine a thin tube, fixed at one end, acted upon by an uniform tangential stress of intensity q.

Let l = length of tube ;
" t = thickness " " ,
" d = diameter " "
then $\pi t d$ = sectional area of tube (approx),
$q\pi t d$ = total shearing force,
$q\pi t d \times \frac{1}{2} d = \frac{1}{2} q\pi t d^2$ = total twisting moment.

This twisting moment is balanced by the resistance at the fixed end. At any transverse section there will be an uniform shearing stress of intensity q.

Let us now consider a very small square on the surface of this tube, with its sides on transverse and longitudinal sections. To balance the stress q on the transverse sides there is called into action an equal stress on the longitudinal sides, so the strength of the tube to resist torsion would be as effectually destroyed if it were cut into longitudinal strips as if cut transversely. But if we make spiral slits at an angle of 45° [S S in fig.], no material being removed, the strength would be unimpaired. The ribbands would be in tension and their sides pressed together. Another set at right angles to these (S' S' in fig.) would be under compression and the tension acting perpendicular to the dividing surfaces would separate them.

CHANGE OF FORM. The square above considered will be changed into a rhombus. A straight line AD on the surface of the tube parallel to the axis will be twisted into a spiral AD', the angle of the spiral being the angle of distortion of the square. Let θ be that angle, then

167 $q = C\theta$ where C is the coëfficient of rigidity
108 The effect of this is that relatively to the end A the end D is twisted through an angle $DOD' = i$ suppose, called the angle of torsion.

In circular measure $i = \frac{\text{arc } DD'}{r}$ but $DD' = l\theta$. θ being a small angle

$$\therefore i = \frac{l\theta}{r} \text{ and, since } \theta = \frac{q}{C},\ i = \frac{ql}{Cr}.$$

TORSION OF A SHAFT [Cotterill § 184 p. 360].

Let us imagine a shaft cut up into a great number of concentric tubes. To the end of each tube a sufficient force is applied to twist it through the same angle i;

$$\text{but } i = \frac{ql}{Cr} = \frac{q_1 l}{Cr_1} = \frac{q_2 l}{Cr_2} \therefore \frac{q}{r} = \frac{q_1}{r_1} = \frac{q_2}{r_2}, \ldots$$

or in words the intensity of the force must be proportional to the distance from the centre. Since each tube is turned through the same angle we may suppose them united again and we have a solid shaft. Hence the shearing stress at any point on a cross-section of the shaft must be proportional to the distance of the point from the centre of the shaft. This is true excepting very near the point of application of the twisting moment.

The total resistance to torsion of the solid shaft is the sum of the twisting moments of the concentric tubes. Thus, $T = \Sigma 2\pi r^2 t q$ in which $q = r\frac{q_1}{r_1}$, when r_1 is the outside radius. Make $t = dr$

$$T = \int 2\pi r^2 dr \cdot r\frac{q_1}{r_1} = \frac{q_1}{r_1}\int 2\pi r^3 dr.$$

The expression $\int 2\pi r^3 dr$ is called the <u>Polar</u> <u>Moment</u> of <u>Inertia</u> and for a circular shaft $\int 2\pi r^3 dr = \frac{1}{2}\pi r^4$,

$$T = \frac{q_1}{r_1} I = \frac{q_1}{r_1} \cdot \frac{\pi r^4}{2} = \frac{1}{2} q_1 \pi r_1^3.$$

Dropping the suffixes and letting r represent the outside radius and f the coefficient of strength of material to resist shearing;

$$T = \frac{1}{2}\pi f r^3 = \frac{1}{16}\pi f d^3.$$

For other than circular shafts the student is referred by our author [C.E. p.493 to the formulæ of M. de St. Venant. Diagrams and particulars with respect to his results may be found in Sir W. Thompson's, Treatise on Natural

p.167 Philosophy, 1st. ed. vol. I p.545.
§.108 The following table given by Cotterill is taken from his
Cont. results.

RELATIVE STRENGTHS OF SHAFTS OF THE SAME SECTIONAL AREA.	
FORM OF SECTION.	STRENGTH.
Circular	1
Square	.8863
Rectangular with sides in the ratio $n:1$	$\sqrt{\frac{2}{n+\frac{1}{n}}} \times .8863$
Ellipse with axes in the ratio $n:1$	$\sqrt{n}$ $(n<1)$

p.168
§.108 ## ELLIPSE OF STRESS. [Cotterill §.199, p. 337].

In simple tension and compression the direction of the stress is the same for all planes, but its intensity varies, becoming zero for planes parallel to the force. In shearing the intensity is the same for all planes perpendicular to a third given plane but the direction varies; on one pair it is normal, on another tangential [page 33 of these notes].

IV <u>Combination of any Two Principal Stresses</u>

PROBLEM. To combine a pair of simple longitudinal stresses the directions of which are at right angles and the intensities given. The stresses may be both tension, both compression, or one tension and one compression.

Let the plane of the paper be parallel to the direction of the stresses and let us consider a piece of material of thickness unity, If the stress be uniform the size and shape of the piece is immaterial.

Let p_1 = greater stress in the direction OX,
" p_2 = lesser " " " " OY.

Let us imagine a rectangular block ABCD with sides perpendicular to the stresses p_1, p_2. On faces AB, CD a stress of intensity p_1 and total amount $p_1 \cdot AB$ will act; on BC, AD there will be a stress of intensity p_2 and total amount $p_2 \cdot BC$. Divide now the rectangle by a diagonal plane AC; there will be a stress on that plane, which it is our object to determine in direction and magnitude.

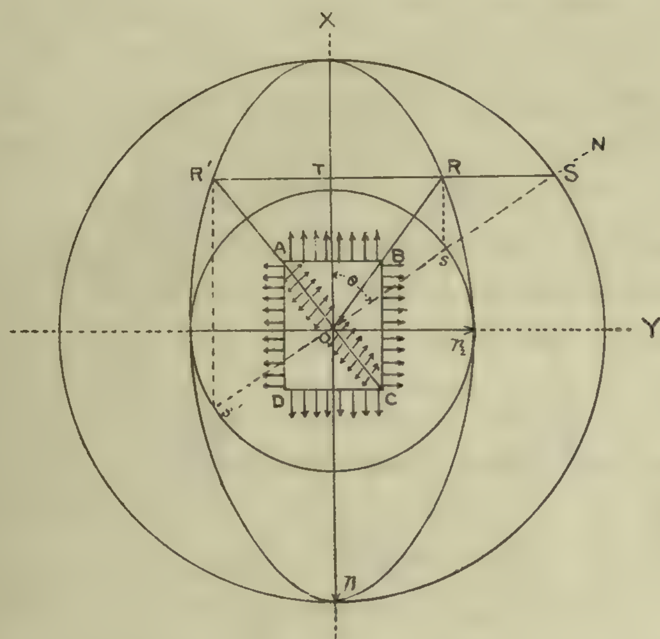
Let $\hat{nx} = \theta$ be the angle which the normal to this plane makes with the axis OX. By changing the length of one

168 or of both sides of this rectangle the angle θ may be made
108 what we please. Proceeding, as on page 31, we find for the normal intensity

$$p_n = p_1 \cos^2\theta + p_2 \sin^2\theta,$$

tangential intensity,

$$p_t = (p_1 - p_2)\sin\theta\cos\theta.$$



The resulting stress might be found in direction and intensity by compounding these results but it is better to proceed by a graphical construction.

On the normal lay off $OS = p_1$ and $Os = p_2$; also draw the ordinates ST parallel to OY and sR parallel to OX, they intersect in R Then

$$OT = OS\cos\theta = p_1 \cdot \frac{AB}{AC}$$

$$RT = Os\sin\theta = p_2 \cdot \frac{BC}{AC}$$

Whence it follows that the intensity of the stress due to p_1 is represented by OT, and that due to p_2 by RT. If then we join OR we shall obtain the resultant stress on AC in direction and magnitude. It will be readily seen that R lies on an ellipse of which p_1 and p_2 are the semi-axes. This ellipse is called the <u>Ellipse</u> of <u>Stress</u>.

If the stresses have opposite signs then $Os = p_2$ must be set off in the opposite direction from OS which gives the point R′.

The locus of the point R may be shown to be an ellipse thus. Solving the equation of the ellipse, $\frac{x^2}{p_1^2} + \frac{y^2}{p_2^2} = 1$, for y we have

$$y = \pm\frac{p_2}{p_1}\sqrt{p_1^2 - x^2}.$$

Now if we describe a concentric circle with a radius p_1 its equation will be $y' = \pm\sqrt{p_1^2 - x^2}$; dividing the 1st. by the 2nd we have

$$\frac{y}{y'} = \frac{p_2}{p_1}.$$

If we refer to the figure we see that this condition is fulfilled since $\frac{TR}{TS} = \frac{Os}{OS}$.

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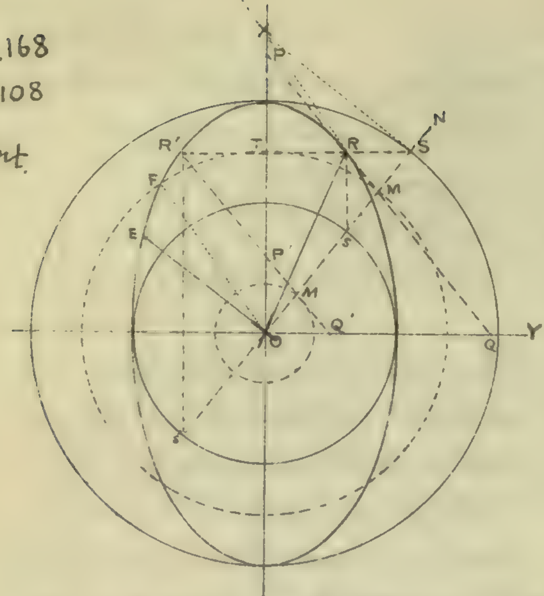
§108
Cont.



It may be seen by the figure that Rankine's construction will give the same point R.

For $OM = QM = MP = \frac{p_1+p_2}{2}$ or the point M is half way between S and s. Since ST and OY are parallel, the triangles OMQ and SMR are similar,

$$\therefore RM = MS = p_1 - \frac{p_1+p_2}{2} = \frac{p_1-p_2}{2},$$

which is the value Rankine makes RM.

$$\text{Since } PR = PM - RM = \frac{p_1+p_2}{2} - \frac{p_1-p_2}{2} = p_2$$

$$\text{and } QR = QM + MR = \frac{p_1+p_2}{2} + \frac{p_1-p_2}{2} = p_1,$$

we see that Rankine's method is identical with one of the usual ways of describing an ellipse, the point P being kept on the axis of X and the point Q on the axis of Y the point R will describe the curve.

We also have $R'M' = s'M' = p_2 + \frac{p_1-p_2}{2} = \frac{p_1+p_2}{2}$, so that by Rankines construction, when one stress is tension and the other compression, that is have opposite signs, we get the same point R' which is given by the intersection of SR' and s'R'.

$$\text{Since } R'Q' = \frac{p_1+p_2}{2} + \frac{p_1-p_2}{2} = p_1$$

$$\text{and } R'P' = \frac{p_1+p_2}{2} - \frac{p_1-p_2}{2} = p_2,$$

when the points P' and Q' are kept on the axes, the point R' will describe the ellipse, a method frequently used.

By reference to either of the last two figures we see that $OR^2 = OT^2 + TR^2$ or $p^2 = p_1^2\cos^2\theta + p_2^2\sin^2\theta$ and since θ is the same as $\hat{nx}$

$$p^2 = p_1^2\cos^2\hat{nx} + p_2^2\sin^2\hat{nx} \quad\ldots\ldots(1)$$

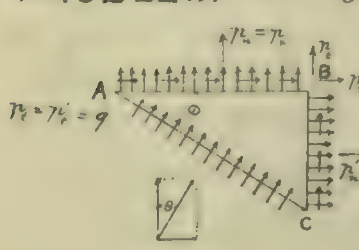
In the triangle RMO, $\sin RMO : \sin\hat{nr} :: OR : RM$,
but the angle $RMO = 180° - (RsM + sRM) = 180° - (\hat{nx} + \hat{nx}) = 180° - 2nx$

$$\therefore \sin RMO = \sin 2\hat{nx} \quad \therefore \sin\hat{nr} = \frac{p_1-p_2}{2p}\cdot\sin 2\hat{nx}$$

$$\therefore \hat{nr} = \text{arc.}\sin\left(\sin 2\hat{nx}\cdot\frac{p_1-p_2}{2p}\right) \quad\ldots\ldots(2)$$

V <u>Deviation</u> of <u>Principal</u> <u>Stresses</u> by a <u>Shearing</u> <u>Stress</u>

PROBLEM This problem as stated by our author is part of the more general problem to show that <u>Any</u> <u>State</u> of <u>Stress</u> in <u>Two</u> <u>Dimensions</u> may always be reduced to a pair of simple stresses such as we have just considered.



Whatever the stresses may be they must

.169 have tangential and normal components on the planes
108 AB and BC, reasoning as on page 32 the tangential components must be equal and opposite.

Let the equal tangential components be of intensity q and the normal components intensities p_x and p_y; which reduces the general problem to the one given in Rankine.

Consider the equilibrium of the triangular portion ABC and let us determine under what conditions it is possible that the stress on AC should be a normal stress only.

If p be that normal stress and we resolve it parallel to BC and AB, for the component parallel to BC, we have

$$p \cdot AC \cdot \cos\theta = q\,BC + p_x AB$$

$$\text{or } p \cdot \frac{AC}{AB} \cdot \cos\theta = q\,\frac{BC}{AB} + p_x = p$$

$$\therefore p - p_x = q \tan\theta \ldots\ldots\ldots (a)$$

Component parallel to AB, in a similar way we get.

$$p - p_y = q \cot\theta \ldots\ldots\ldots (b)$$

Multiplying equation (a) by eq. (b) we get

$$(p - p_x)(p - p_y) = q^2\,;$$

the roots of this equation are

$$\left.\begin{aligned} p_1 &= \frac{p_x + p_y}{2} + \sqrt{\frac{(p_x - p_y)^2}{4} + q^2} \\ p_2 &= \frac{p_x + p_y}{2} - \sqrt{\frac{(p_x - p_y)^2}{4} + q^2}\,. \end{aligned}\right\} \ldots\ldots\ldots (3)$$

Subtracting equation (a) from (b),

$$p_x - p_y = q(\cot\theta - \tan\theta) = 2q \cdot \cot 2\theta \quad \text{[C.E. p.41 eq. 12]}$$

$$\therefore \tan 2\theta = \frac{2q}{p_x - p_y}\,; \text{ or } \cot 2\theta = \frac{p_x - p_y}{2q} \ldots\ldots\ldots (4)$$

This equation always gives two values of θ showing that two planes at right angles can always be drawn on which the stress is wholly normal. The maximum value of θ is 45° when $p_x = p_y$.

Having found p_1 and p_2 the <u>Ellipse</u> of <u>Stress</u> can be constructed.

Conjugate stresses are not represented by conjugate diameters as might naturally be supposed. In the figure on p.38, OR and OE represent conjugate stresses, while OF is the diameter conjugate to OR.

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Cont.

FIG. A

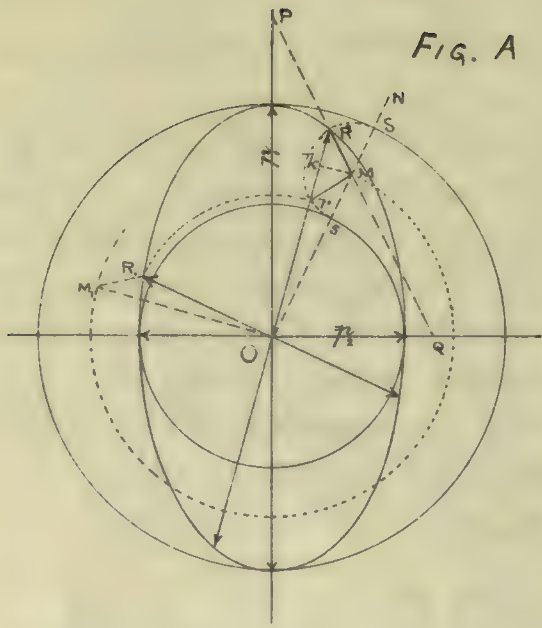
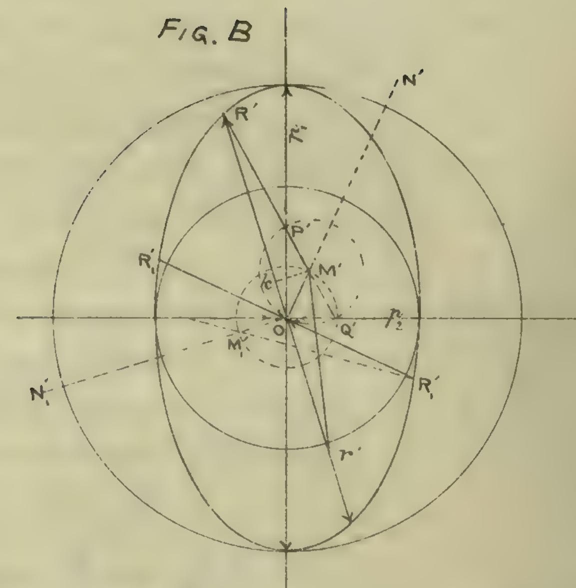


FIG. B



PROBLEM *Given two conjugate stresses to find graphically the principal stresses.*

When the stresses are of the same kind, FIG. A, as OR and OR_1. Take Or = OR_1, bisect Rr at k, draw kM ⊥ to OR, connect M and R and produce the line each way making MQ = MP = MO; P and Q are points on the principal axes and give the directions while p_1 = RQ, p_2 = RP.

Remembering that conjugate stresses have a common obliquity we see that when one is placed upon the other the normal lines coincide and the point M must be on the normal and equally distant from R and r; hence the construction.

When the stresses are of opposite kind the construction is as shown in FIG. B; Or′ = OR_1', P′ and Q′ are points on the axes, p_1 = R′Q′, p_2 = R′P′

<u>FLUID</u> <u>PRESSURE</u>.

If the stresses are of the same kind and equal intensity on planes at right angles to each other, the ellipse reduces to a circle. From equation (1) p.38, if $p_1 = p_2$, we have $p = p_1 = p_2$. And from eq. (2) the angle $NOR = \hat{nr} = 0$. Therefore the stress is the same intensity on all planes, and the direction is normal.

<u>OPPOSITE</u> <u>AND</u> <u>EQUAL</u> <u>PRINCIPAL</u> <u>STRESSES</u>.

In equation (1) if $p_1 = -p_2$ then, $\pm p = p_1 = -p_2$; and from eq.(2) $\hat{nr} = 2\hat{nx}$. Therefore the stress on any plane is of the same

p.169 intensity and the angles which its direction makes with the
§108 normal to its plane are bisected by the axes of principal stress.
cont.

FRICTION.

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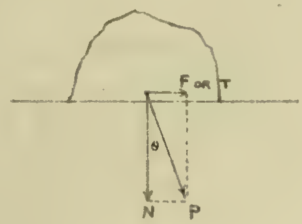

$$F = fN. \quad \text{.............} (1)$$

$N = P\cos\theta,$

$T = P\sin\theta \;=\; N\tan\theta.$

For stability $\quad T \leqq fN \quad \therefore$

$$\frac{N}{T}(=\tan\theta) \leqq f.$$

The arc. tan f is called the angle of repose and is denoted by φ.
If P = the weight of a body and φ is the inclination of an inclined plane on which the body will just slide, then, remembering that $\sin\varphi = \dfrac{\tan\varphi}{\sqrt{1+\tan^2\varphi}}$, we have

$$F = T = fN = N\tan\varphi = P\sin\varphi = \frac{fP}{\sqrt{1+f^2}} \quad \text{......} (2)$$

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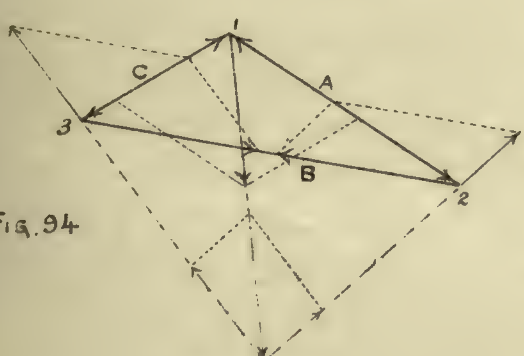

FIG. 94

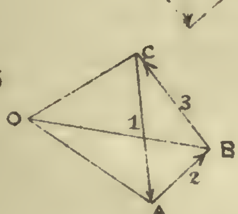

FIG. 95

It must be remembered in assuming the triangle of forces ABC that the forces when applied to the frame must have their lines of action to meet in one point [unless they are parallel], otherwise they will not be in equilibrium, and the lines from A, B, C [fig. 95] will not meet in one point O.

In the fig. (94) the forces which act at each joint are shown and it is easy to see the corresponding lines in the polygon of forces.

Each piece of the frame exerts equal forces in opposite directions.

To find the direction and intensity of the force exerted by any piece of the frame as A. In the polygon of forces the triangle ABO represents the forces acting at 2 (FIG. 94); the arrow head on AB gives us the direction in which we go around the triangle so the piece A exerts a force in the direction from O to A = OA; ∴ A is a strut.

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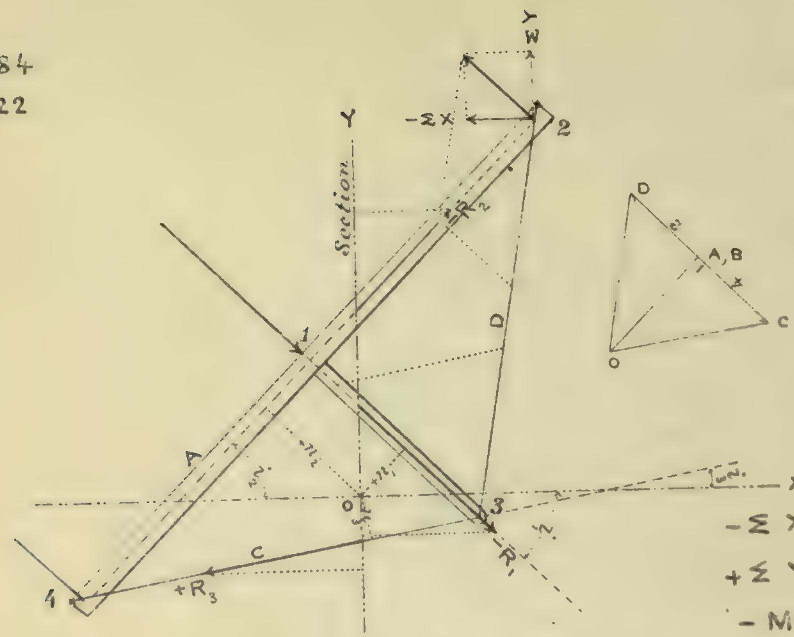

The frame is cut by a plane whose trace is OY, R_1, R_2 are thrusts $\therefore -$, R_3 is tension $\therefore +$; n_1, n_2 are upwards $\therefore +$, n_3 is downward $\therefore -$. Observing these signs and writing equations (1) we have

$$\left.\begin{aligned} -\Sigma X &= -R_1 \cos i_1 - R_2 \cos i_2 + R_3 \cos i_3 \\ +\Sigma Y &= +R_1 \sin i_1 - R_2 \sin i_2 + R_3 \sin i_3 \\ -M &= -R_1 n_1 - R_2 n_2 - R_3 n_3 . \end{aligned}\right\} (1)$$

Since the values of $R_1 \cos i_1$, $R_1 \sin i_1$ etc. are given in the drawing the student should verify the first two of the above equations

It must be remembered that in using the method of sections the signs of R_1, R_2, R_3 are not known until equations (1) are solved.

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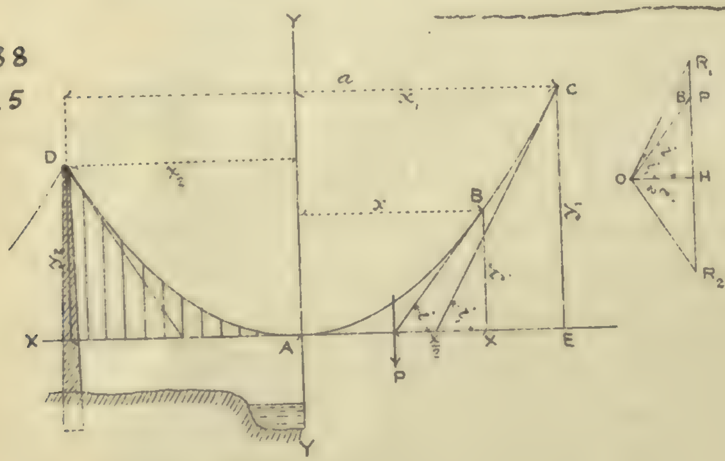

SUSPENSION BRIDGE.

$$P = px,$$

$$\tan i = \frac{dy}{dx} = \frac{P}{H} = \frac{px}{H} ;$$

$$\therefore y = \int \frac{p}{H} x \, dx = \frac{p}{H} \int x \, dx ,$$

$$\therefore y_0 = \frac{p}{H} \frac{x^2}{2} \text{ and } x^2 = \frac{2H}{p} y :$$

but $x^2 = 4my$ where m is the focal distance,

$$\therefore m = \frac{x^2}{4y} = \frac{H}{2p} \quad \ldots\ldots\ldots (3.)$$

PROBLEM I. Given y_1, y_2, a; to find x_1, x_2, m.

For the point C, $x_1^2 = 4my_1$; for D $x_2^2 = 4my_2$

$$\therefore x_1 = \sqrt{4m} \cdot \sqrt{y_1} ; \quad x_2 = \sqrt{4m} \cdot \sqrt{y_2}$$

$$\therefore a = x_1 + x_2 = \sqrt{4m}\,(\sqrt{y_1} + \sqrt{y_2}) \quad \therefore \sqrt{4m} = \frac{a}{\sqrt{y_1} + \sqrt{y_2}}$$

hence we have

$$x_1 = \frac{a\sqrt{y_1}}{\sqrt{y_1} + \sqrt{y_2}} ; \quad x_2 = \frac{a\sqrt{y_2}}{\sqrt{y_1} + \sqrt{y_2}} \quad \ldots\ldots\ldots (4.)$$

$$\sqrt{4m} = \frac{a}{\sqrt{y_1} + \sqrt{y_2}} \quad \therefore m = \frac{a^2}{4(\sqrt{y_1} + \sqrt{y_2})^2} = \frac{a^2}{4y_1 + 4y_2 + 8\sqrt{y_1} \cdot \sqrt{y_2}} \quad \ldots\ldots\ldots (5.)$$

p 189 When $y_1 = y_2$ we have $x_1 = x_2 = \frac{a}{2}$; $m = \frac{a^2}{16y_1}$ .... (6.)

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Sept.

PROBLEM II. In the equation, $\tan i_1 = \frac{2y_1}{x_1}$, substitute the value of x_1 from eq. (4)

PROBLEM III. From (3) $H = 2pm$, substituting for m its value from (5) we get
$$H = 2pm = \frac{pa^2}{2y_1 + 2y_2 + 4\sqrt{y_1 y_2}} \quad\text{..................(9.)}$$

$$R_1 = H \sec i_1 = H\sqrt{1+\tan^2 i_1} = H\sqrt{\left(1+\frac{4y_1^2}{x_1^2}\right)} \;;\quad R_2 = H\sqrt{\left(1+\frac{4y_2^2}{x_2^2}\right)} \quad\text{..............(10.)}$$

If $y_1 = y_2$, from (6), $m = \frac{a^2}{16y_1}$; $H = 2pm = \frac{pa^2}{8y_1}$; $R_1 = R_2 = H\sqrt{\left(1+\frac{16y_1^2}{a^2}\right.}$ ...... (11.)

PROBLEM IV. To find the length of the curve.

$$x^2 = 4my \;\therefore\; \frac{dy}{dx} = \frac{x}{2m}.$$

calculus

$$\left\{s = \int\left(1+\frac{dy^2}{dx^2}\right)^{1/2} dx\right\} \therefore s = \int\left(1+\frac{x^2}{4m^2}\right)^{\frac{1}{2}} dx = \frac{1}{2m}\int(4m^2+x^2)^{\frac{1}{2}} dx \quad\text{.............(a.)}$$

$$\text{Let } (4m^2+x^2)^{\frac{1}{2}} = z - x \quad\text{.......................(b.)}$$

$$s = \frac{1}{2m}\left[\int z\,dx - \int x\,dx\right] = -\frac{x^2}{4m} + \frac{1}{2m}\int z\,dx \quad\text{..............................(c)}$$

$$\text{From (b)} \quad 4m^2 + x^2 = z^2 - 2zx + x^2.$$

$$\therefore x = \frac{z^2-4m^2}{2z} = \frac{z}{2} - \frac{2m^2}{z} \text{ and } dx = \frac{dz}{2} + \frac{4m^2\,dz}{2z^2};$$

substituting this value of dx in the expression $\int z\,dx$ we have
$$\int z\,dx = \frac{1}{2}\int z\,dz + 2m^2\int\frac{dz}{z} = \frac{1}{4}z^2 + 2m^2\log z,$$
$$\text{from (b) } z = x + \sqrt{(4m^2+x^2)},$$
$$\therefore z^2 = x^2 + 2x\sqrt{(4m^2+x^2)} + 4m^2 + x^2;$$

$$\therefore s = \frac{1}{2m}\left[-\frac{x^2}{2} + \frac{x^2}{2} + \frac{x\sqrt{4m^2+x^2}}{2} + m^2 + 2m^2\log.\left[x+\sqrt{4m^2+x^2}\right)\right. + C$$
$$= \frac{m}{2} + \frac{x\sqrt{4m^2+x^2}}{4m} + m\log.\left[x+\sqrt{4m^2+x^2}\right] + C$$

Beginning at A, the origen, we have $s = 0$, $x = 0$
$$0 = \frac{m}{2} + m\log.2m + C \;\therefore\; C = -\frac{m}{2} - m\log.2m.$$

$$\therefore s = \frac{x\sqrt{4m^2+x^2}}{4m} + m\log\frac{x+\sqrt{4m^2+x^2}}{2m} \;;\quad\text{...........................(d)}$$

which may be written
$$s = \frac{x}{2m}\cdot\frac{2m}{2}\sqrt{1+\frac{x^2}{4m^2}} + m\log.\left[\frac{x}{2m}+\sqrt{1+\frac{x^2}{4m^2}}\right]$$

Since $x^2 = 4my$, $\frac{dy}{dx} = \frac{x}{2m} = \tan i$, and $\sqrt{1+\frac{x^2}{4m^2}} = \sec i$.

$$\therefore s = m\left[\tan i \sec i + \text{hyp. log.}(\tan i + \sec i)\right] \quad\text{...........(12)}$$

p.190 §125 *Cont.* The equation, in the text, immediately preceding (12) may be derived from (d) by substituting for m its value $\frac{x^2}{4y}$.

$$s = \int\sqrt{\left(1+\frac{dy^2}{dx^2}\right)}\,dx = \int\left(1+\frac{x^2}{4m^2}\right)^{\frac{1}{2}}dx = \int\left(1+\frac{x^2}{4\,\frac{x_1^4}{16y_1^2}}\right)^{\frac{1}{2}} = \int\left(1+\frac{4y_1^2x^2}{x_1^4}\right)^{\frac{1}{2}}dx\,,$$

developing by the Binomial Theorem,

$$s = \int\left[1+\tfrac{1}{2}\cdot\frac{4y_1^2x^2}{x_1^4}+\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\cdot\frac{16y_1^4x^4}{x_1^8}+\ldots\ldots\right]dx$$

$$= \int dx\left(1+\frac{2y_1^2x^2}{x_1^4}-\frac{4y_1^4x^4}{x_1^8}+\text{etc.}\ldots\ldots\right]$$

Since the ratio $\frac{y_1}{x_1}$ is small for spans where the cable is stretched tightly we may neglect the term involving $\frac{y_1^4}{x_1^8}$ which leaves

$$s = \int dx\left[1+\frac{2y_1^2x^2}{x_1^4}\right] = \int dx + \frac{2y_1^2}{x_1^4}\int x^2dx = x + \frac{2}{3}\frac{y_1^2x^3}{x_1^4} + C\ ;$$

beginning at the origen, and making $\frac{y_1^2}{x_1^4} = \frac{y^2}{x_4}$, we have

$$s = x + \frac{2}{3}\frac{y^2}{x}, \text{nearly}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13.)$$

PROBLEM V. It must be remembered that the points C and D are fixed while the origen A moves; $y_1 = y_2 +$ Constant; $\therefore\ dy_1 = dy_2 = dy$. x_1 and x_2 change very little and are considered constant for this slight motion.

$$\text{From eq. (14)}\quad s_1 + s_2 = a + \frac{2}{3}\left(\frac{y_1^2}{x_1}+\frac{y_2^2}{x_2}\right)$$

$$\therefore\ d(s_1+s_2) = \frac{4}{3}\frac{y_1}{x_1}dy_1 + \frac{4}{3}\frac{y_2}{x_2}dy_2 \quad \therefore\ \frac{d(s_1+s_2)}{dy} = \frac{4}{3}\left(\frac{y_1}{x_1}+\frac{y_2}{x_2}\right)\ldots\ldots\ldots\ldots(16.)$$

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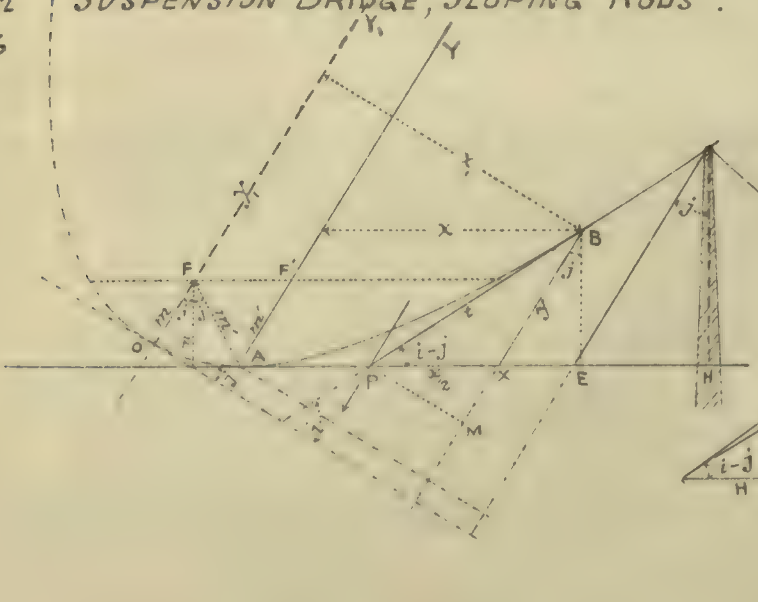
p. 192 §126 SUSPENSION BRIDGE, SLOPING RODS.

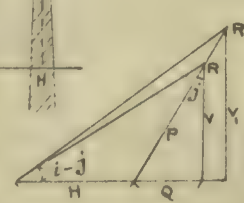
m = focal distance principal axes;
m' = " " conj. diam. "



From the figure we have

$$m' = n\sec j\,,$$

$$n = m\sec j\ ;$$



$$\therefore\ m' = m\sec^2 j$$

$$= \frac{m}{\cos^2 j}$$

$$\therefore\ y = \frac{x^2\cos^2 j}{4m}\ \ldots\ldots(4)$$

193 $H+Q=\left(\frac{P}{2y}+q\right)x = \frac{2vx^3\sec j}{4y} + vx\tan j$; from eq. (5) $\frac{x^2}{4y} = \frac{m}{\cos^2 j}$

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$$\therefore H+Q = v\left(2m\sec^3 j + x\tan j\right) \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$\cos i = \frac{PN}{PB} = \frac{\frac{1}{2}x\cos j}{t} = \frac{x}{2t}\cos j \quad \therefore\ i = \text{arc.}\cos\left(\frac{x}{2t}\cdot\cos j\right) \ldots\ldots\ldots\ldots (11)$$

From the figure it may be seen that j = the inclination of the curve at the point A to a perpendicular to the axis $\therefore$ if we take eq. (12), § 125, p 190, between the limits j and i we have

$$\text{parabolic arc. } AB = s = m\left\{\tan i\sec i - \tan j\sec j + \text{hyp.}\log\frac{\tan i+\sec i}{\tan j+\sec j}\right\} (12)$$

The approximate formula is obtained by substituting in eq. (13) § 125, p 190, for x the distance $AE = x + y\sin j$

" y " " $BE = y\cos j$

$$\therefore \text{arc } AB = s = x + y\sin j + \frac{2}{3}\frac{y^2\cos^2 j}{x+y\sin j} \ldots\ldots\ldots\ldots\ldots\ldots (13.)$$

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194 *DEFLECTION OF A FLEXIBLE TIE*

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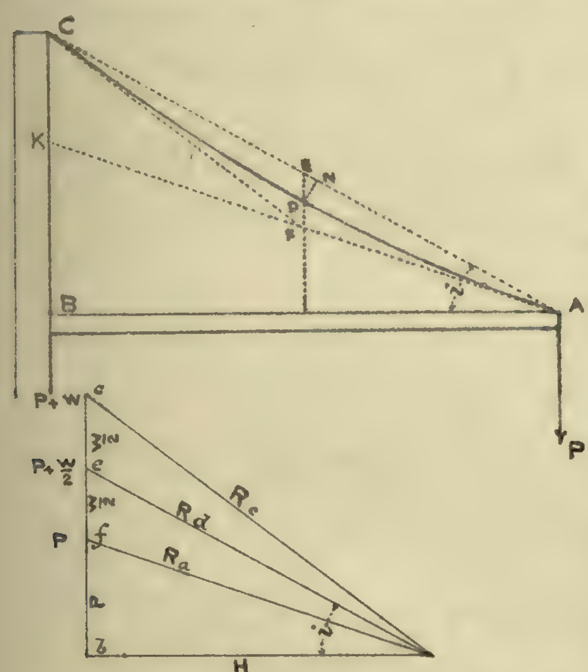

$H = \left(P+\frac{W}{2}\right)\frac{1}{\tan i}$; $\tan i = \frac{be}{bo} = \frac{BC}{AB}$ $\therefore$

$$\left.\begin{aligned} &H = \left(P+\tfrac{W}{2}\right)\frac{AB}{BC} \quad ; \quad R_a = \sqrt{H^2+P^2} \; ; \\ &R_d = \sqrt{H^2+\left(P+\tfrac{W}{2}\right)^2} \; ; \quad R_e = \sqrt{H^2+(P+W)^2}. \end{aligned}\right\} (1)$$

$DE = \frac{1}{2}EF = \frac{1}{4}CK = \frac{1}{4}(BC - BK)$

$BK : BC :: bf : be :: P : P+\frac{W}{2}$ $\therefore$

$BK = BC\frac{P}{P+\frac{W}{2}}$ $\therefore$ $DE = \frac{1}{4}\left(BC - BC\frac{P}{P+\frac{W}{2}}\right)$ $\therefore$

$DE = \frac{1}{4}BC\left(1 - \frac{P}{P+\frac{W}{2}}\right) = \frac{1}{4}BC\left(\frac{P+\frac{W}{2}-P}{P+\frac{W}{2}}\right)$ $\therefore$

$$DE = \frac{1}{4}BC\left(\frac{\frac{W}{2}}{P+\frac{W}{2}}\right) = \frac{1}{8}BC\frac{W}{P+\frac{W}{2}} \ldots\ldots (1)$$

$\left\{\text{Eq. (15) § 125, p. 190.} \quad 2s_1 = a + \frac{8}{3}\frac{y_1^2}{a}\right\}$

$2s_1 = ADC$; $a = AC$; $y = DN$.

$DN : DE :: AB : AC$ $\therefore$

$DN = \frac{AB\cdot DE}{AC} = y_1$ $\therefore$

$ADC = AC + \frac{8}{3}\frac{\left(\frac{AB\cdot DE}{AC}\right)^2}{AC}$ $\therefore$

$ADC - AC = \frac{8}{3}\frac{AB^2 DE^2}{AC^3}$ $\therefore$ from eq. 1

$ADC - AC = \frac{8}{3}\frac{AB^2\cdot\frac{1}{64}BC^2\cdot\left(\frac{W}{P+\frac{W}{2}}\right)^2}{AC^3}$ $\therefore$

$$ADC - AC = \frac{1}{24}\cdot\frac{AB^2\cdot BC^2}{AC^3}\left(\frac{W}{P+\frac{W}{2}}\right)^2 \ldots\ldots (2)$$

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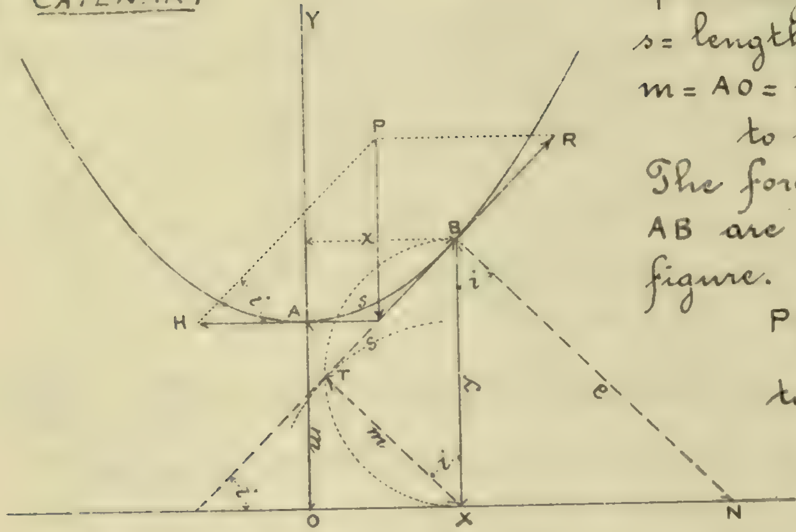

p = weight of a units length of chain,
s = length of chain from A to B
m = AO = length of ch. equal in wt. to the horizontal pull H.

The forces acting on the portion AB are P, R, H, as shown in the figure.

$$P = ps \ ; \quad H = pm \ ;$$

$$\tan i = \sqrt{\sec^2 i - 1} = \sqrt{\frac{ds^2}{dx^2} - 1}$$

$$= \frac{P}{H} = \frac{ps}{pm} = \frac{s}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (a.)$$

$$\therefore \frac{ds^2}{dx^2} = 1 - \frac{s^2}{m^2} \quad \therefore \frac{ds}{dx} = \frac{\sqrt{s^2+m^2}}{m} \quad \therefore \quad \frac{dx}{m} = \frac{ds}{\sqrt{m^2+s^2}}; \ldots\ldots\ldots\ldots (b.)$$

to integrate this place $\sqrt{m^2+s^2} = z - s$ $\ldots\ldots\ldots\ldots\ldots\ldots$ (c.)

$$m^2+s^2 = z^2 - 2zs + s^2 \ \therefore s = \frac{z^2-m^2}{2z} = \frac{z}{2} - \frac{m^2}{2z} \ \therefore ds = \frac{dz}{2} + \frac{m^2\,dz}{2z^2} = \frac{z^2+m^2}{2z^2}\,dz$$

$$\therefore \frac{ds}{\sqrt{(m^2+s^2)}} = \frac{\frac{z^2+m^2}{2z^2}\cdot dz}{z - \frac{z^2-m^2}{2z}} = \frac{\frac{z^2+m^2}{2z^2}\,dz}{\frac{z^2+m^2}{2z}} = \frac{dz}{z} \quad \therefore$$

$$\int \frac{dx}{m} = \int \frac{ds}{\sqrt{(m^2+s^2)}} \quad \therefore \frac{x}{m} = \int \frac{dz}{z} = \log z + C$$

$\therefore \frac{x}{m} = \log(s + \sqrt{m^2+s^2} + C$. When $x = 0$, $s = 0$; $\therefore C = -\log m$

which gives $\frac{x}{m} = \log \frac{s+\sqrt{m^2+s^2}}{m}$ $\ldots\ldots\ldots\ldots\ldots\ldots$ (d.)

$$\cot i = \sqrt{\operatorname{cosec}^2 i - 1} = \sqrt{\frac{ds^2}{dy^2} - 1} = \frac{H}{P} = \frac{pm}{ps} = \frac{m}{s} \quad \therefore \frac{ds^2}{dy^2} - 1 = \frac{m^2}{s^2}$$

$$\frac{ds}{dy} = \frac{\sqrt{s^2+m^2}}{s} \quad \therefore dy = \frac{s\,ds}{\sqrt{(m^2+s^2)}}$$

$\therefore y = \int (m^2+s^2)^{-\frac{1}{2}} s\,ds = (m^2+s^2)^{\frac{1}{2}} + C$. When $y = m$, $s = 0$ $\therefore C = 0$

$$\therefore y^2 = m^2 + s^2 \text{ and } s = \sqrt{y^2 - m^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (e.)$$

From equation (d), $\frac{x}{m} = \log \frac{\sqrt{y^2-m^2}+y}{m}$ $\ldots\ldots\ldots\ldots\ldots\ldots$ (1)3rd

$$\therefore \varepsilon^{\frac{x}{m}} = \frac{y}{m} + \frac{\sqrt{y^2-m^2}}{m} ; \text{ or } \varepsilon^{\frac{x}{m}} - \frac{y}{m} = \frac{\sqrt{y^2-m^2}}{m}$$

squaring $$\varepsilon^{\frac{2x}{m}} - 2\varepsilon^{\frac{x}{m}}\frac{y}{m} + \frac{y^2}{m^2} = \frac{y^2}{m^2} - 1$$

$$\therefore y = \frac{\varepsilon^{\frac{2x}{m}} + 1}{2\varepsilon^{\frac{x}{m}}}\, m$$

$$\text{or} \quad y = \frac{m}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right) \ldots\ldots\ldots\ldots\ldots\ldots (1)^{\text{st}}$$

196 from eq. (e) $y = \sqrt{m^2+s^2}$ ∴ $y = \sqrt{m^2+s^2} = \frac{m}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)$

28 ∴ $4\,\frac{m^2+s^2}{m^2} = \varepsilon^{\frac{2x}{m}} + 2 + \varepsilon^{-\frac{2x}{m}}$ ∴ $4 + \frac{4s^2}{m^2} = \varepsilon^{\frac{2x}{m}} + 2 + \varepsilon^{-\frac{2x}{m}}$

$$\therefore \frac{4s^2}{m^2} = \varepsilon^{\frac{2x}{m}} - 2 + \varepsilon^{-\frac{2x}{m}}$$

$$\therefore s = \frac{m}{2}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right) = \sqrt{y^2-m^2} \quad \ldots\ldots (1.)^{2d.}$$

$$\text{Area} = \int y\,dx = \frac{m}{2}\int\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right) m\,\frac{dx}{m} = \frac{m^2}{2}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right) = ms \quad \ldots\ldots (1.)^{4th.}$$

$$\frac{dy}{dx} = \tan i = \frac{P}{H} = \frac{ps}{pm} = \frac{s}{m} = \frac{1}{2}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right) \quad \ldots\ldots (1.)^{5th.}$$

$\left\{\rho = \dfrac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\right\}$; but $\frac{dy}{dx} = \frac{s}{m}$ ∴ $\frac{d^2y}{dx^2} = \frac{1}{m}\frac{ds}{dx}$;

and since $s = \frac{m}{2}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right)$, $ds = \frac{m}{2}\left(\varepsilon^{\frac{x}{m}}\frac{dx}{m} + \varepsilon^{-\frac{x}{m}}\frac{dx}{m}\right)$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{2m}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right) = \frac{y}{m^2}$$

$$\therefore \rho = \frac{\left(1+\frac{s^2}{m^2}\right)^{\frac{3}{2}}}{\frac{y}{m^2}} = \frac{\left(\frac{y^2}{m^2}\right)^{\frac{3}{2}}}{\frac{y}{m^2}} = \frac{\frac{y^3}{m^3}}{\frac{y}{m^2}} = \frac{y^2}{m} = \frac{m}{4}\left(\varepsilon^{\frac{2x}{m}} + 2 + \varepsilon^{-\frac{2x}{m}}\right) \quad \ldots\ldots (f.)$$

At the point A, $y = m$ ∴ $\rho_1 = m$.
Let $XT = OA = m$, then $BT^2 = BX^2 - XT^2 = y^2 - m^2 = s^2$; $\frac{BT}{TX} = \frac{s}{m} = \frac{dy}{dx}$.

$BN = BX.\sec XBN = y \sec i = y.\frac{BN}{BX}$; from similar triangles BXT and BNX we have,

$$\frac{BN}{BX} = \frac{BX}{BT} \;\therefore\; BN = \frac{BX^2}{XT} = \frac{y^2}{m} = \rho \quad \ldots\ldots (g.)$$

again at B, $R = \sqrt{H^2+P^2} = p\left(\sqrt{m^2+s^2}\right) = py$ ......(2.)

197 IV PROBLEM

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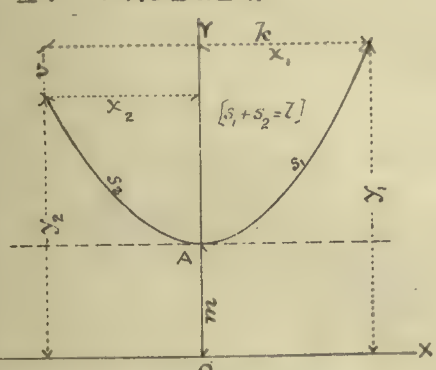
nt.



N.B. y_1 and y_2 had better be measured from OX than from AX' though equations (3) and (4) will be the same in either case.

Let $x_1 - x_2 = h$, an unknown quantity;
$x_1 + x_2 = k$, then $x_1 = \frac{k+h}{2}$; $x_2 = \frac{k-h}{2}$.

$$s_1 + s_2 = l = \frac{m}{2}\left(\varepsilon^{\frac{k+h}{2m}} - \varepsilon^{-\frac{k+h}{2m}} + \varepsilon^{\frac{k-h}{2m}} - \varepsilon^{-\frac{k-h}{2m}}\right);$$

factoring, $l = \frac{m}{2}\left(\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right)\left(\varepsilon^{\frac{h}{2m}} + \varepsilon^{-\frac{h}{2m}}\right)$ ......(h);

also, $y_1 - y_2 = v = \frac{m}{2}\left(\varepsilon^{\frac{k+h}{2m}} + \varepsilon^{-\frac{k+h}{2m}} - \varepsilon^{\frac{k-h}{2m}} - \varepsilon^{-\frac{k-h}{2m}}\right)$;

and $v = \frac{m}{2}\left(\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right)\left(\varepsilon^{\frac{h}{2m}} - \varepsilon^{-\frac{h}{2m}}\right)$ ......(i.)

$$\therefore l^2 = \frac{m^2}{4}\left(\varepsilon^{\frac{k}{m}} - 2 + \varepsilon^{-\frac{k}{m}}\right)\left(\varepsilon^{\frac{h}{m}} + 2 + \varepsilon^{-\frac{h}{m}}\right)$$

$$v^2 = \frac{m^2}{4}\left(\varepsilon^{\frac{h}{m}} - 2 + \varepsilon^{-\frac{k}{m}}\right)\left(\varepsilon^{\frac{h}{m}} - 2 + \varepsilon^{-\frac{h}{m}}\right)$$

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Sup.

By subtraction $l^2 - v^2 = \frac{m^2}{4}\left(\varepsilon^{\frac{k}{m}} - 2 + \varepsilon^{-\frac{k}{m}}\right)\cdot 4$

$$\sqrt{l^2 - v^2} = m\left(\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right) \quad \dots\dots\dots\dots \quad (3.)$$

Adding eqs. (h) & (i) $\quad l + v = \frac{m}{2}\left(\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right). 2\,\varepsilon^{\frac{h}{2m}}$,

subtracting, $\quad l - v = \frac{m}{2}\left(\varepsilon^{\frac{k}{2m}} - \varepsilon^{-\frac{k}{2m}}\right). 2\,\varepsilon^{-\frac{h}{2m}}$

$$\frac{l+v}{l-v} = \frac{\varepsilon^{\frac{h}{2m}}}{\varepsilon^{-\frac{h}{2m}}} = \varepsilon^{\frac{h}{m}} \quad \therefore \frac{h}{m} = \log.\frac{l+v}{l-v} \quad \therefore h = m \log.\frac{l+v}{l-v},$$

whence

$$\left.\begin{aligned} x_1 &= \tfrac{1}{2}\left(m \text{ hyp. log.}\frac{l+v}{l-v} + k\right) \\ x_2 &= -\tfrac{1}{2}\left(m \text{ hyp. log.}\frac{l+v}{l-v} - k\right) \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (4.)$$

To compare the catenary with the parabola having a focal distance $\frac{m}{2}$, expand the equation of the catenary $y = \frac{m}{2}\left[\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right]$ by McLaurin's Theorem

in which $\quad y = (y) + \left(\frac{dy}{dx}\right)\frac{x}{1} + \left(\frac{d^2y}{dx^2}\right)\frac{x^2}{1.2} + \left(\frac{d^3y}{dx^3}\right)\cdot\frac{x^3}{1.2.3} + \text{etc.}$

When $x = 0 \quad (y) = \frac{m}{2}\cdot 2 = m$

$$\left(\frac{dy}{dx}\right) = \frac{m}{2}\cdot\left(\frac{1}{m}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right)\right) = 0 \qquad \left(\frac{d^3y}{dx^3}\right) = \frac{m}{2m^2}\cdot\frac{1}{m}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right) = 0$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{m}{2m}\cdot\frac{1}{m}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right) = m\cdot\frac{1}{m^2} \qquad \left(\frac{d^4y}{dx^4}\right) = \frac{m}{2m^3}\cdot\frac{1}{m}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right) = m\cdot\frac{1}{m^4}.$$

Hence $\quad y = m\left[1 + \frac{x^2}{2m^2} + \frac{x^4}{24m^4} + \frac{x^6}{720m^6} + \text{etc.}\dots\right] \quad \dots\dots\dots\dots \quad (k)$

The equation to the parabola whose focal distance $= \frac{1}{2}m$ is

$$x^2 = 2my,$$

the origen being at the vertex. Change the origen to the point O at a distance $= m$ below the vertex, this being the origen for the catenary; we have

$$y - m = \frac{x^2}{2m} \quad \therefore y = m\left(1 + \frac{x^2}{2m^2}\right) \quad \dots\dots\dots\dots \quad (l)$$

To get the length of the catenary, $\frac{dy}{dx} = \frac{s}{m} \quad \therefore s = \frac{1}{m}\cdot\frac{dy}{dx}$,

find value $\frac{dy}{dx}$ from eq. (k). $\quad \therefore s = x\left[1 + \frac{x^2}{6m^2} + \frac{x^4}{120m^4} + \text{etc.}\dots\right]$

For the parabola $s = \int\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}.dx$; from eq. (l) $\frac{dy}{dx} = \frac{x}{m} \therefore$

$s = \int\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}\,dx = \int\sqrt{\left(1 + \frac{x^2}{m^2}\right)}\,dx = \frac{1}{m}\int\sqrt{m^2 + x^2}\cdot dx$. Expand $(m^2 + x^2)^{\frac{1}{2}}$ by the binomial theorem & integrate, which gives eq. in text.

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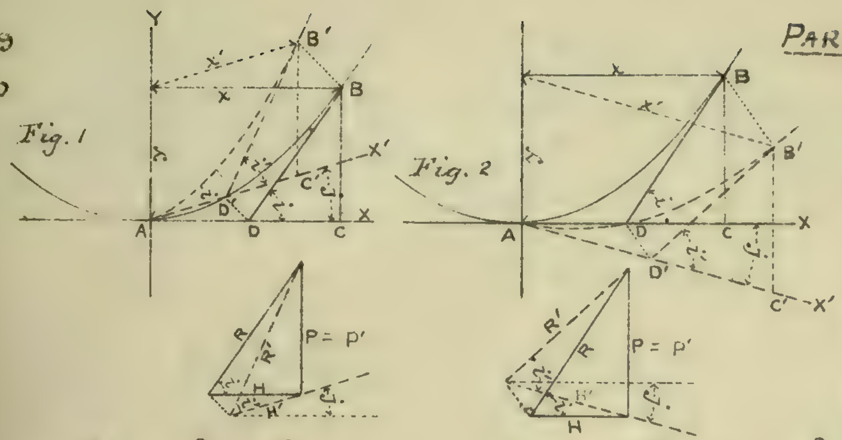
§130



PARALLEL PROJECTIONS

The full lines refer to the original curve and the dotted lines to the parallel projection.
The ordinate y remains the same for both while x is changed to x' which is oblique and shorter in fig. 1, but longer in fig. 2. From the triangles in the figures or from the diagrams of forces all the equations given in the text may be written. e.g.

$$P' : H' : R' :: \sin \widehat{H'R'} : \sin \widehat{P'R'} : \sin \widehat{P'H'}$$
$$:: \sin i' \;:\; \cos(i \pm j) : \cos j \quad \dots\dots\dots\dots (5.)$$

TRANSFORMED CATENARY

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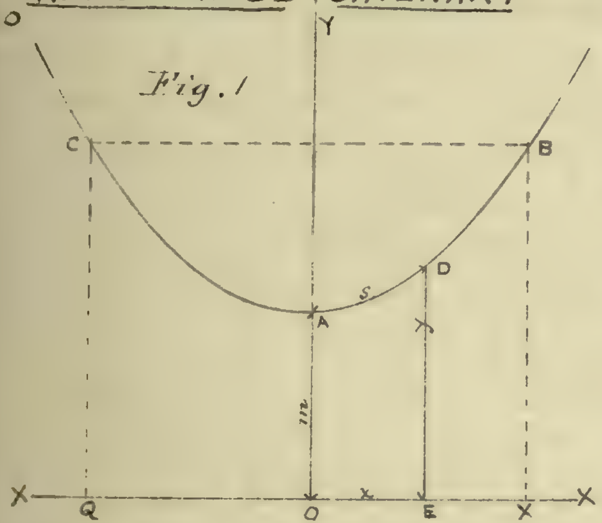
§131



In the common catenary (Fig. 1) since the area AOED = $m s$ and m is constant the area varies as s. But the load on the arc AD (= ps) also varies as s, p being constant. Hence a convenient mode of representing the load on any arc AD is to suppose a sheet of metal CQXBAC, bounded below by the "directrix" QX, to be suspended from the curve.

Let the weight of this metal corresponding to m units of its surface = p; that is, $wm = p$ or $w = \frac{p}{m}$.
The weight of a strip a unit in breadth extending from A to O is then = p = wt. of a units length of the chain.
Then the part of the sheet AODE whose wt. = $wms = ps$ represents the wt. P on the arc AD. So AOBX represents the wt. on AB and CQXB the whole wt. on CAB.

For the horizontal force at A we have.

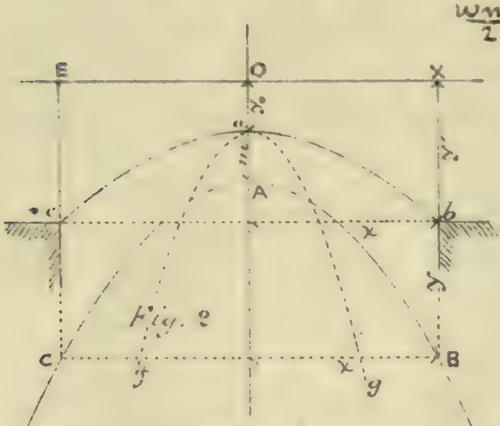
$$H = pm = wm^2 \dots\dots\dots\dots (1)$$

At any point D, $\quad T = \sqrt{H^2 + P^2} = p\sqrt{s^2 + m^2} = py = wmy \dots\dots (a)$

☞ Up to this point we have used p to mean the same thing as in §.128 viz. the load pr. units length of curve = $\frac{dP}{ds}$, but

p.201 in this article (§ 131) our author uses p to mean the load per
§131 units leath in the direction of the axis OX $= \frac{dP}{dx}$.
Cont. The load between A and B $= P = w \cdot OAXB = w\int y\,dx = \frac{wm^2}{2}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right)$; ...(1)^{3rd}
since $P = w\int y\,dx$ $\quad \frac{dP}{dx} = wy = p = \frac{wm}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)$; ....(1.)^{2nd}

In what follows ∴ we must write wm for p in using the equations of § 128.
Thus we have, the tension at B $= \sqrt{P^2+H^2} = wm\sqrt{m^2+s^2} = wmy =$
$\frac{wm^2}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)$ .... (1.)^{4th}



It makes no change in the form of the curve AB to increase or diminish the weights provided the proportion among them is preserved. Note, however, that we cannot change the depth AO of the sheet without changing the curve. Hence the modulus (m = AO) fixes the catenary; or, if we assume the catenary the modulus is determined. This often interferes with the use of the common catenary in the building of arches in which case it is inverted as shown in Fig. 2. The metal sheet is replaced by a wall of uniform material and the tension on the chain is replaced by a thrust along the rib.

We are often compelled to make a curve pass through three points, as a, b, & c Fig. 2, while it is evident that if we assume Oa = the modulus we get the curve f a g which does not pass through the points c & b.

To overcome this difficulty we use the transformed catenary as described in the text; the values of x and H remain unchanged while the values of y and P are changed in the ratio

$$m : y_0 :: y : y' :: P : P'$$

$$\therefore\ y = \frac{my'}{y_0}, \quad \text{and} \quad P = \frac{m}{y_0}P'. \qquad \left[\text{In the figure } \frac{m}{y_0} = 2\right]$$

From the 1st. of Eqs. (1) § 128 p. 196 $\quad \left\{y = \frac{m}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)\right\}$ ∴

$$\frac{m}{y_0}y' = \frac{m}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right) \quad \therefore \quad y' = \frac{y_0}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right); \text{ .... (a)}$$

the load between a & b $= P' = w\int y'\,dx = \frac{wmy_0}{2}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right)$; .... (3)
the intensity at b $= p' = \frac{dP'}{dx} = wy' = \frac{wy_0}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)$; .... (3)^{2nd}
the tension at b $= T = \sqrt{P'^2 + H^2}$; $\quad H' = H$; $P' = \frac{y_0}{m}P$ ....(3)^{4th}

p.201 In the above equations the value of m is as yet unknown,
§131 to find it we have Eqs. (1)[3rd] §128, p. 196.

$$\left\{ x = m \log\left(\frac{y}{m} + \sqrt{\frac{y^2}{m^2} - 1}\right) \right\} \text{ but } \frac{y}{m} = \frac{y'}{y_0} \therefore$$

$$m = \frac{x}{\log.\left(\frac{y'}{y_0} + \sqrt{\frac{y'^2}{y_0^2} - 1}\right)} \quad \text{.........................} \quad (4)$$

Knowing m gives the catenary CAB which transformed gives the curve cab.

To find the inclination of the curve at the point b we have $\tan i = \frac{P'}{H} = \frac{dy'}{dx} = \frac{y_0}{2m}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right)$.

$$\left\{ \rho = \frac{\left(1 + \frac{dy'^2}{dx^2}\right)^{3/2}}{\frac{d^2y'}{dx^2}} \right\} \quad \therefore \rho = \frac{\left[1 + \frac{y_0^2}{4m^2}\left(\varepsilon^{\frac{x}{m}} - \varepsilon^{-\frac{x}{m}}\right)^2\right]^{3/2}}{\frac{y_0}{2m^2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)}.$$

EXAMPLE Given the span of an arch = 9 ft. (= $2x$); rise = 4½ ft. [= $y' - y_0$]; height of brickwork over the crown = 24½ ft. (= y_0); w = 112 lbs., wt. per cubic foot. Determine the transformed catenary and find the amount and the direction of the thrust at the abutments.

Answer, $i = 64°\ 05'$; H = 6359 lbs; P = 13083 lbs.

The transformed catenary is the theoretical curve to support a wall, as there is no horizontal component to the pressure, but it would not do to support an earth embankment.

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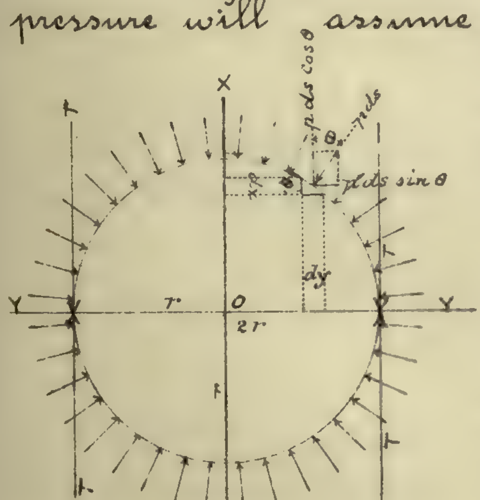
p.203 *CIRCULAR RIB FOR FLUID PRESSURE.*

§133 A flexible tube when acted on by an uniform internal pressure will assume a circular cross-section, or the curvature will be uniform, and to resist an uniform pressure from without the curvature must also be uniform. The circle is the only curve which fulfills this condition therefore the linear arch must be circular.



The pressure on $ds = p\,ds$, the vertical component $= p \cdot ds \cdot \cos\theta$, this is distributed over the surface $dy = ds \cdot \cos\theta$,

$$\therefore p_x = \frac{p \cdot ds \cdot \cos\theta}{ds \cdot \cos\theta} = p = \frac{p\, ds \sin\theta}{ds \sin\theta} = p_y.$$

p.204 $2T = \int_{-r}^{+r} p_x\,dy = p_x \int_{-r}^{+r} dy = 2p_x r \quad \therefore T = p_x r = pr = H = P$ .... (1) & (3)

§133 It is evident without integration that the resultant

Int. pressure on a semi-cylinder must be the same as on a flat surface equal to the diameter, each being a unit in length. If P = pressure at either end of the diameter then total pressure $= 2P = 2pr$.

For fluid pressure see p.40 of these Notes.

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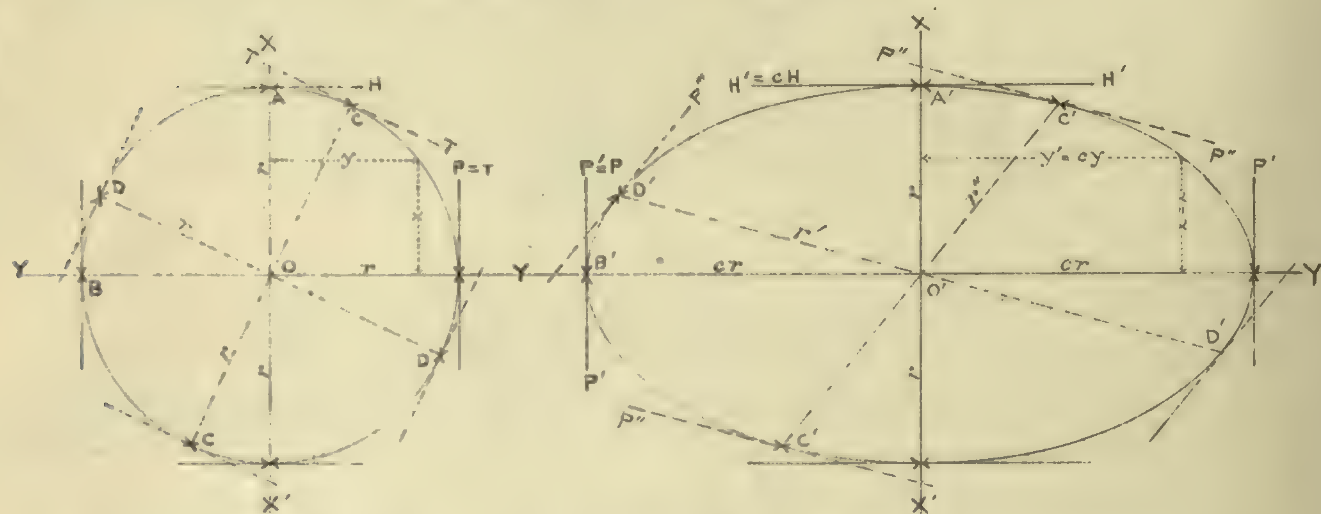
§134



p = intensity of pressure in original circle,

$$x' = x \;;\; y' = cy \quad \text{.......... (1.)}$$

$$P' = P = T = pr \;;\quad H' = cH = cT = cpr \quad \text{.......... (2.)}$$

$$\therefore \frac{P'}{H'} = \frac{pr}{c \cdot pr} = \frac{r}{cr}$$

$$P'' : P :: r' : r \quad \therefore P'' = P\frac{r'}{r} = \frac{p r \cdot r'}{r} = pr' \quad [\text{also } P''' = pr''] \quad \text{.......... (3.)}$$

$$p'_x = \frac{P'}{cr} = \frac{p}{c} \;;\; p'_y = \frac{H'}{r} = \frac{cH}{r} = \frac{c \cdot pr}{r} = cp \quad \text{.......... (4.)}$$

$$\therefore \frac{p'_x}{p'_y} = \frac{1}{c^2} = \frac{r^2}{c^2 r^2} \quad \therefore \frac{OB'}{OA'} = c = \sqrt{\frac{p'_y}{p'_x}} \quad \text{.......... (5.)}$$

Since the direction of the pressure is normal to the circle a line representing it will pass through the centre; consequently the parallel projection of this line passes through the centre of the ellipse.

$$p' = \frac{P''}{r''} \text{ but, eq.(3), } P'' = pr' \quad \therefore p' = p\frac{r'}{r''} \;;\; p'' = \frac{P'''}{r'} = p\frac{r''}{r'} \quad \text{.......... (6.)}$$

$$\therefore \frac{p'}{p''} = \frac{r'^2}{r''^2} \quad \therefore p' : p'' :: r'^2 : r''^2 \quad \text{.......... (7.)}$$

Rankine's notation, which we follow, is confusing in that he uses P'' for the force in the direction of r', whose intensity is p'.

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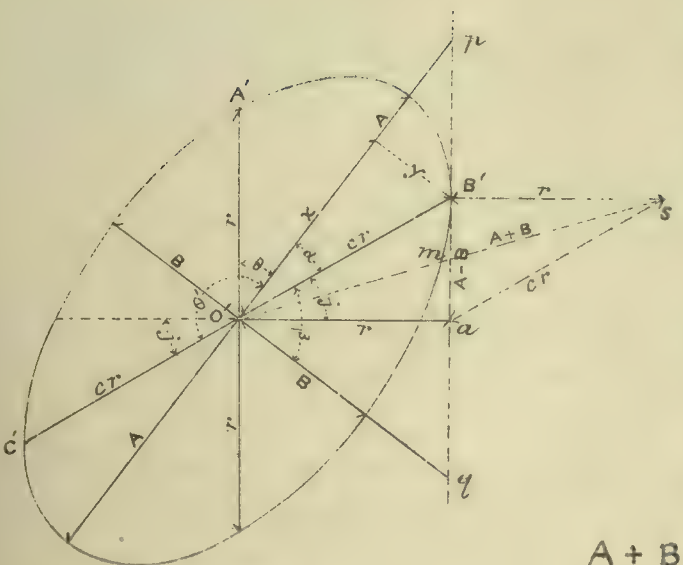
§ 135



To find equations (5) & (6).
Given $O'A' = r$; $O'B' = cr$ conjugate diameters to find the principal axes A and B. and the angles α and β.

Eqs.(5) may be found both analytically and also from the graphical construction thus proving the latter to be a correct method.

From the triangle O'B'S

$$A + B = \sqrt{(r^2 + c^2r^2 + 2r\cdot cr\cos j)}; \;\ldots\ldots (a)$$

from the triangle O'B'a

$$A - B = \sqrt{(r^2 + c^2r^2 - 2r\cdot cr\cos j)}; \;\ldots\ldots (b)$$

$$\left.\begin{aligned} \therefore\quad A &= \frac{A+B}{2} + \frac{A-B}{2} = \frac{r}{2}\left\{\sqrt{(1+c^2+2c\cos j)} + \sqrt{(1+c^2-2c\cos j)}\right\}; \\ B &= \frac{A+B}{2} - \frac{A-B}{2} = \frac{r}{2}\left\{\sqrt{(1+c^2+2c\cos j)} - \sqrt{(1+c^2-2c\cos j)}\right\}; \end{aligned}\right\}\ldots(5)$$

From Analyt. Geom. $\left\{A'B'\sin(\theta'-\theta) = AB\ ;\ A'^2 + B'^2 = A^2 + B^2\right\}$;
from the fig. it may be seen that $\theta' - \theta = C'O'A' = 90 + j \;\therefore \sin(\theta'-\theta) = \cos j$;
$A' = r$; $B' = cr$ $\quad\therefore\quad r\cdot cr\cos j = AB,$

$$r^2 + c^2r^2 = A^2 + B^2;$$

doubling the first, adding and subtracting, we get eqs. (a) & (b) above.

To find α and β. $\left\{\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1\right\}$; but $\left\{\begin{array}{l} x = cr\cos\alpha \\ y = cr\sin\alpha \end{array}\right.$ $\therefore$

$$\frac{1}{c^2r^2} = \frac{\cos^2\alpha}{A^2} + \frac{\sin^2\alpha}{B^2} = \frac{1}{A^2} - \frac{\sin^2\alpha}{A^2} + \frac{\sin^2\alpha}{B^2}, \;\therefore\; \frac{A^2 - c^2r^2}{A^2\cdot c^2r^2} = \sin^2\alpha\,\frac{A^2 - B^2}{A^2B^2};$$

$$\therefore \sin\alpha = \frac{B}{cr}\sqrt{\frac{A^2 - c^2r^2}{A^2 - B^2}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

When $cr > r$ then $\alpha < \beta$; but when $cr < r$ then $\alpha > \beta$.
$\beta = 90 - \alpha \quad\therefore \sin\beta = \cos\alpha$

$$\frac{1}{c^2r^2} = \frac{\cos^2\alpha}{A^2} + \frac{1 - \cos^2\alpha}{B^2} \quad\therefore\ \cos\alpha = \sin\beta = \frac{A}{cr}\sqrt{\frac{B^2 - c^2r^2}{B^2 - A^2}} \;\ldots\ldots\ldots (6.)$$

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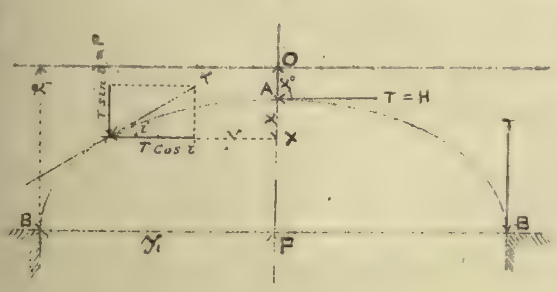
§ 136



HYDROSTATIC ARCH.

For fluid pressure $p = p_x = p_y = wx$.

$$P = \int_0^y p_y\,dy = w\int_0^y x\,dy = T\sin i \;\ldots\ldots (6.)$$

$$H = \int_{x_0}^{x_1} p_x\,dx = w\int_{x_0}^{x_1} x\,dx = w\frac{x_1^2 - x_0^2}{2} = T = p\varrho \;\ldots (7.)$$

$$\int_x^{x_1} p_x\,dx = w\int_x^{x_1} x\,dx = w\,\frac{x_1^2 - x^2}{2} = T\cos i \;\ldots (9.)$$

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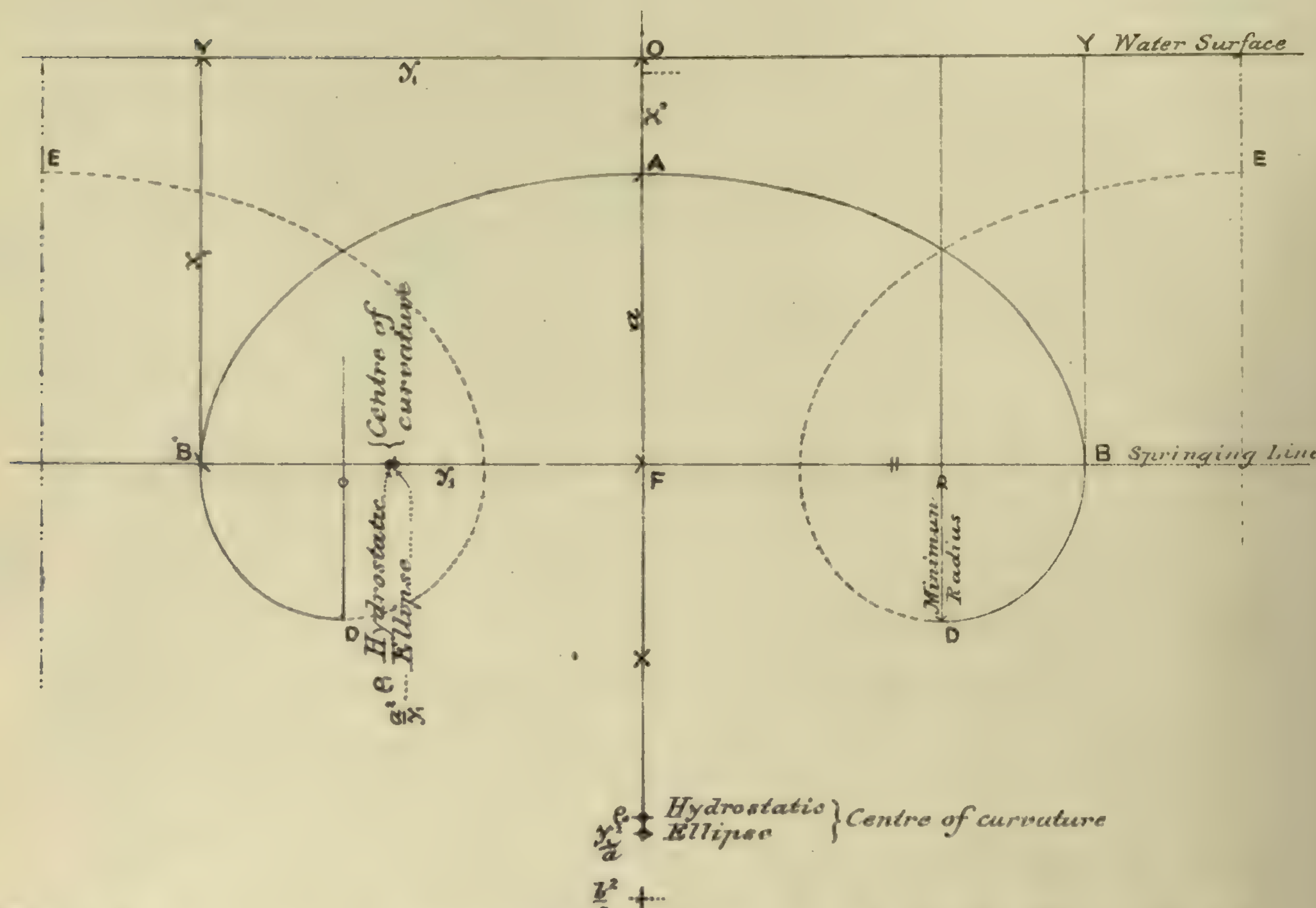

The above curve is accurately drawn and differs so little from the ellipse whose semiaxes are a and y_1 that the curves coincide as nearly as can be drawn. The centres of curvature for each at the points A and B are shown.

PROBLEM

Assuming that the radii of curvature at A & B are in the same ratio as the radii of curvature of an ellipse whose semi-axes are a & $b = y_1 + \dfrac{y_1^2}{30a}$ ....(11.)

$$\rho_0 x_0 = \rho_1 x_1 \quad \therefore \quad \frac{x_1}{x_0} = \frac{\rho_0}{\rho_1} = \frac{\frac{b^2}{a}}{\frac{a^2}{b}} = \frac{b^3}{a^3} \quad \therefore \quad x_1 = x_0 \frac{b^3}{a^3}$$

but [see fig.] $x_1 - x_0 = a \quad \therefore \quad x_0\left(\dfrac{b^3}{a^3} - 1\right) = a \quad \therefore \quad x_0 = a \cdot \dfrac{a^3}{b^3 - a^3}$ ......(11.)

From (10) $\left\{ x_0 \rho_0 = \dfrac{x_1^2 - x_0^2}{2} \right\} \therefore \quad \rho_0 = \dfrac{x_1^2 - x_0^2}{2x_0} = \dfrac{x_0^2 + 2ax_0 + a^2 - x_0^2}{2x_0} \quad \therefore$

$$\rho_0 = \frac{2ax_0 + a^2}{2x_0} = a + \frac{a^2}{2x_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12.)$$

$$\rho_1 = \frac{x_1^2 - x_0^2}{2x_1} = \frac{x_1^2 - x_1^2 + 2ax_1 - a^2}{2x_1} = \frac{2ax_1 - a^2}{2x_1} = a - \frac{a^2}{2x_1} = a - \frac{a^2}{2(x_0 + a)} \quad \ldots\ldots \quad (12)$$

N.B. $b = y_1$ is a closer approximation in the curve drawn above than $b = y_1 + \dfrac{y_1^2}{30a}$.

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GEOSTATIC ARCH.

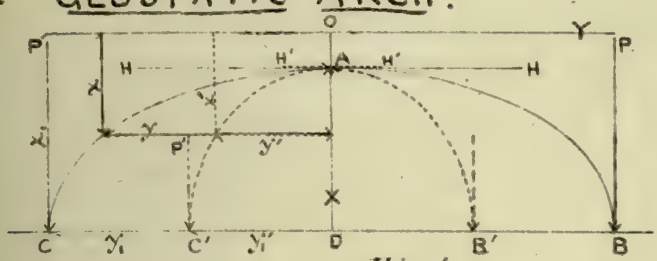

Fig. 1

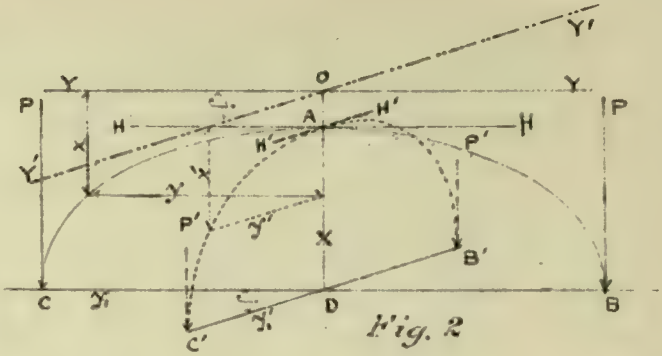

Fig. 2

Fig 1 shows projection when the conjugate plane is horizontal.
Fig. 2 " " " " " " sloping.

FIG. 1. We have $x' = x$; $y' = cy$; $dy' = c\,dy$ ....(a)
P' = weight of material above the arch between C' and A
if $w' = w$ then $P' = w\int_0^{y_1'} x\,dy'$; but $dy' = c\,dy$, and $\int_0^{y_1} wx\,dy = P$;

$$\therefore P' = c\cdot w\int_0^{y_1} x\,dy = cP \text{ .... (b)}$$

If as is usual in parallel projections P, which is parallel to the unchanged ordinate x, had remained unchanged then H' would have $= cP'$ and also $= cP = cH$; since P' is in the above $= cP$ we have

$$H' = cP' = c^2P = c^2H \text{ .... (c)}$$

$p_x = wx = p$; $p_y = \frac{dH'}{dx}$ $\therefore H' = \int_0^{x_1} p_y\,dx$ $\therefore c^2H = c^2P = \int_0^{x_1} p_y\,dx$;

$\therefore H = \frac{1}{c^2}\int_0^{h} p_y\,dx$; but $P = H = \int_0^{x_1} p_x\,dx = \frac{1}{c^2}\int_0^{x_1} p_y\,dx$ $\therefore \frac{p_y}{p_x} = c^2$

$\left[\begin{array}{l}\frac{1}{c^2}\int p_y\,dx = \int p_x\,dx \\ \frac{1}{c^2}p_y\,dx = p_x\,dx \therefore \frac{1}{c^2}p_y = p_x\end{array}\right]$ $\therefore c = \sqrt{\frac{p_y}{p_x}}$ .... (1.)

The above is applicable to Fig. 1, only, which is the simpler of the two: to modify it to suit Fig. 2 we have

FIG. 2

$$P' = w\int_0^{y_1'} x\,dy'\cos j \text{ , but } dy' = c\,dy$$

$$\therefore P' = c\cos j\, w\int_0^{y_1} x\,dy = c\cos j\cdot P \text{ .... (2.)}$$

$$H' = cP' = c^2\cos j\cdot P \text{ .... (2.)}$$

$p_x = wx = p$; $\frac{p_y}{p_x} = \frac{H'}{H} = \frac{c^2\cos j\,P}{P} = c^2\cos j$ $\therefore c = \sqrt{\frac{p_y}{p_x\cos j}}$ .... (1')

If $j = 0$, $\cos j = 1$ so the equations of Fig. 2 are the general ones of which the others are special cases. Consequently eq. (1) in the text should be written as eq. (1') above.

At A and C' Fig. 1 the pressure is normal and we can find the radius of curvature at each.

AT CROWN, $H' = p_0\rho_0' = c^2H = c^2p_0\rho_0$ $\therefore \rho_0' = c^2\rho_0$;

p 213 §137 cont. AT SPRINGING $P' = p_{1y}t_1' = cP = cp_1\rho_1$; but $\{p_y = c^2 p\}$

$$\therefore p_{1y} = c^2 p_1 \quad \therefore c^2 p_1 \rho_1' = c p_1 \rho_1$$

$$\therefore \rho_1' = \frac{1}{c}\rho_1 .$$

N. B. The above relations between radii of curvature do not hold when the conjugate plane is sloping, a fact not mentioned in the text.

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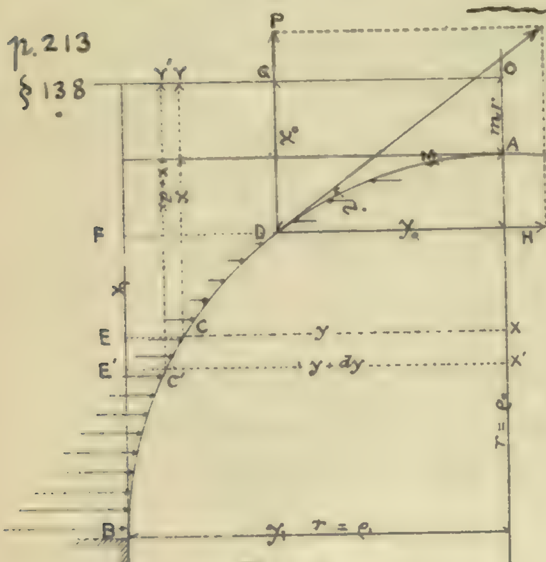

Fig. 1

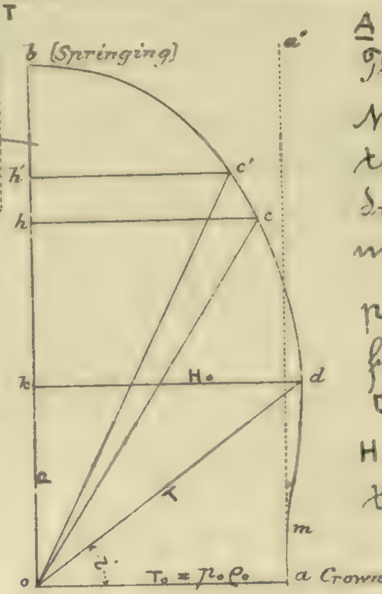

Fig. 2

A LINEAR ARCH OR RIB.
The accompanying figures, Nos. 1 & 2, while illustrating the general discussion are drawn to suit Example IV under Problem V, where $p_x = wx$; $p_{x_0} = wmr$; and to find the forces acting at D we have $P = w \cdot$ Area ADGO, H is horizontal and T is in the direction of the tangent at D, hence we can construct the parallelogram of forces as shown in Fig. 1 from which we can write,

$$\left.\begin{aligned} H &= P\frac{dy}{dx} = P\cot i\,; \\ T &= \sqrt{P^2+H^2} = P\operatorname{cosec} i \end{aligned}\right\} \cdots (1)$$

It must be borne in mind that at A there is no vertical component, and the Area ADGO has reduced to zero $\therefore P = 0$. Similarly at B there is no horizontal component $\therefore H = 0$.

Beginning at B the horizontal component of the thrust increases to D where it reaches a maximum [see Fig. 2] and is equal to H_0. As may be seen from the

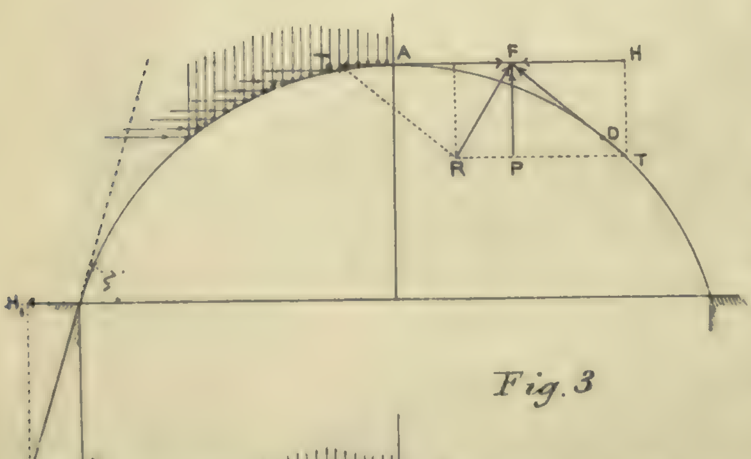

Fig. 3

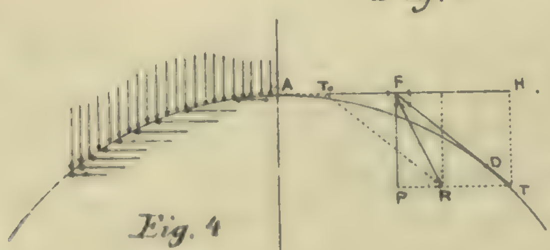

Fig. 4

curve b c' c d, Fig. 2, the intensity is greatest near b [B]. From B (b) to D (d) the spandril must push horizontally

p. 214 §138 Cont. against the rib, between D (d) and M it must pull horizontally and finally between M and A it practically neither pulls nor pushes. The arrow heads represent the horizontal pressure while the vertical pressure is represented by vertical lines terminating in OY.

The forces acting on any are as AD in Figs. 3 & 4 may be drawn as shown. In Fig 3 the thrust at the crown, T_o, is greater than the horizontal component at D, H; or $T_o = H + RP$ $\therefore$ $RP = T_o - P \cdot \cot i$ = the push which the spandril must give the rib between D and A. In Fig. 4 $T_o = H - RP$ $\therefore$ $-RP = T_o - P \cdot \cot i$ = pull between A & D.

The line FR = R represents the resultant of all external forces acting on the portion of the rib AD.

PROBLEM V may be stated thus, *To find the position of the resultant of the horizontal thrusts below the point of maximum thrust.*

In Fig. 1 the problem is to find the position of the resultant of the horizontal thrusts between D and B; H_o is the sum of all these forces and the point of application is found by making the moments about O equal.

If the arch rises obliquely as from C then the part of H_o which consists of the horizontal component of the thrust at the springing, $P_1 \cot i$, has the lever arm x_1.

$$\therefore x_H = \frac{\int_0^{H_o} x\,dH}{H_o} = \frac{\int_{x_o}^{x_1} x\,p_y\,dx + x_1 P_1 \cot i_1}{H_o} \quad \text{.......... (5)}$$

<u>Example I</u>. Catenary &c. $p_y = 0$; $H_o = P_1 \cot i_1 = \text{constant}$ $\therefore$ $x_H = x_1$ (6)

<u>Example II</u>. Semicircular Rib, uniform normal pressure.

$p_x = p_y = p = \text{constant}$; $H_o = pr$; $\int_{x_o}^{x_1} x\,p_y\,dx = p\int_{x_o}^{x_1} x\,dx = \frac{1}{2}pr^2$; $\cot i_1 = 0$

$$\therefore x_H = \frac{\frac{1}{2}pr^2}{pr} = \frac{r}{2} \quad \text{.......... (7)}$$

<u>Example III</u> Hydrostatic Arch. $p_y = wx$; $H_o = \int_{x_o}^{x_1} wx\,dx = w\frac{x_1^2 - x_o^2}{2}$

$$\int_{x_o}^{x_1} x\,p_y\,dx = w\int_{x_o}^{x_1} x^2\,dx = \frac{x_1^3 - x_o^3}{3} \quad \therefore x_H = \frac{2}{3} \cdot \frac{x_1^3 - x_o^3}{x_1^2 - x_o^2} \quad \text{.......... (8)}$$

<u>Example IV</u> Figures 1 and 2 illustrate this example, they are drawn to scale with considerable accuracy and show the way the forces act.

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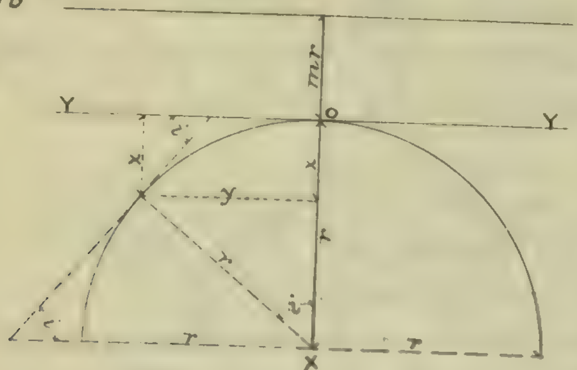
Cont.

Example IV, continued.



$$x = r(1-\cos i)\;;\quad dx = r\sin i\,di\;;$$
$$y = r\sin i\;;\quad dy = r\cos i\,di\;;$$
$$T_0 = p_0\rho_0 = wmr\cdot r = wmr^2.$$
$$P = w\int(mr+x)\,dy = w\int\left[mr + r(1-\cos i)\right]\times r\cos i\,di,$$
$$= wr^2\int(m+1-\cos i)\cos i\,di,$$
$$= wr^2\left[\int(m+1)\cos i\,di - \int\cos^2 i\,di\right]$$

but $\int\cos^2 i\,di = \int(1-\sin^2 i)\,di = i - \int\sin^2 i\,di;$

$$\left\{\int u\,dv = uv - \int v\,du\right\};\quad u = \sin i\;;\quad du = \cos i\,di\;;\quad dv = \sin i\,di\;;\quad v = -\cos i\;;$$
$$\therefore \int\sin^2 i\,di = -\sin i\cos i + \int\cos^2 i\,di,$$
$$\therefore \int\cos^2 i\,di = i + \sin i\cos i - \int\cos^2 i\,di,$$
$$\therefore \int\cos^2 i\,di = \frac{i}{2} + \frac{\sin i\cos i}{2}.$$
$$\therefore P = wr^2\left\{(1+m)\sin i - \frac{\sin i\cos i}{2} - \frac{i}{2}\right\} \quad\ldots\ldots\ldots\ldots(a)$$

From eq.(1) $\left\{H = P\cot i = P\dfrac{\cos i}{\sin i}\right\}$.

$$H = wr^2\left\{(1+m)\sin i - \frac{\sin i\cos i}{2} - \frac{i}{2}\right\}\frac{\cos i}{\sin i} = wr^2\left\{(1+m)\cos i - \frac{\cos^2 i}{2} - \frac{i}{2}\cdot\frac{\cos i}{\sin i}\right\}.$$
$$\therefore H_0 = wr^2\left\{(1+m)\cos i_0 - \frac{\cos^2 i_0}{2} - \frac{i_0\cot i_0}{2}\right\} \quad\ldots\ldots (11.)$$
$$p_y = -\frac{dH}{dx} = -\frac{wr}{\sin i}\frac{d}{di}\left\{(1+m)\cos i - \frac{\cos^2 i}{2} - \frac{i}{2}\cot i\right\}$$
$$\frac{d(\cos^2 i)}{2\sin i\,di} = \frac{-\cos i\sin i}{\sin i} = -\cos i$$
$$\frac{d(i\cot i)}{\sin i\,di} = \frac{\cot i\,di - i\sec i\,di}{2\sin i\,di} = \frac{\cos i}{2\sin^2 i} - \frac{i}{\sin^3 i} = -\frac{i - \sin i\cos i}{2\sin^3 i}$$
$$\therefore p_y = wr\left(1 + m - \cos i - \frac{i - \sin i\cos i}{\sin^3 i}\right) \quad\ldots\ldots\ldots\ldots (9.)$$
$$p_y = 0\;;\quad i = \text{arc.}\cos\left\{1 + m - \frac{i - \sin i\cos i}{\sin^3 i}\right\}\;;$$

the empirical formula used for the first approximation is

$$i = \text{arc.}\cos\frac{1+3m}{2}\quad\text{approximate}\quad\ldots\ldots\ldots (10.)$$

From eq.(5) p.216 $\left\{x_H = \dfrac{\int_{x_0}^{x_1} x\,p_y\,dx + x_1 P_1\cot i_1}{H_0}\right\}$; $i_1 = 90°\ \therefore\ \cot i_1 = 0$

$$\therefore\quad x_H = \frac{1}{H_0}\int_{x_0}^{x_1} x\,p_y\,dx = \frac{1}{H_0}\int_{i_0}^{i_1} r(1-\cos i)\,p_y\cdot r\sin i\,di$$
$$\therefore\ x_H = \frac{r^2}{H_0}\int_{i_0}^{90°} p_y\sin i\,(1-\cos i)\,di \quad\ldots\ldots\ldots\ldots (12.)$$

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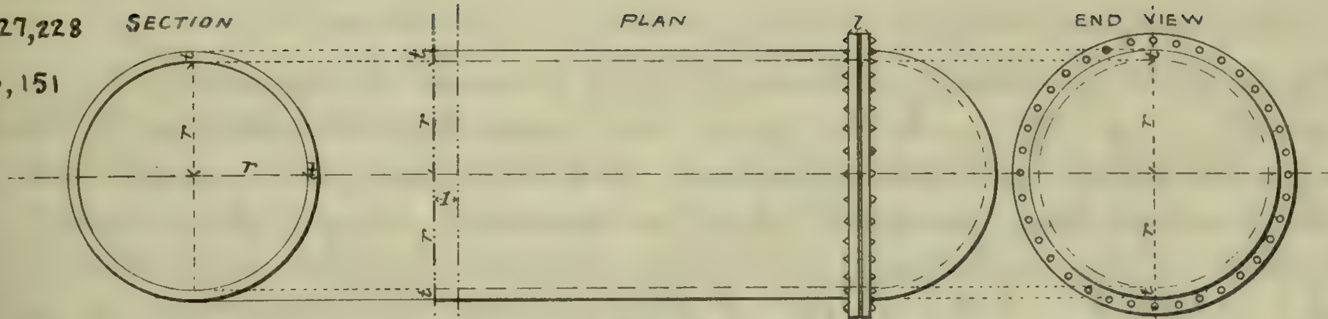

Fig. 1

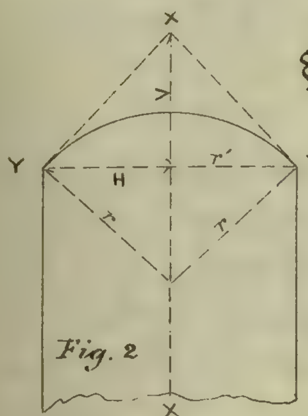

Fig. 2

§150 — $T = pr = ft \therefore p = \frac{ft}{r} \therefore \frac{t}{r} = \frac{p}{f}$ .... (1), (2.)

§ 151 $P = \pi r^2 p$; $S = 2\pi r t$ nearly $\therefore$

$$f = \frac{P}{S} = \frac{\pi r^2 p}{2\pi r t} = \frac{pr}{2t} \quad \text{.......} \quad (1')$$

$$\frac{t}{r} = \frac{p}{2f} \quad \text{.......} \quad (2')$$

Fig. 2 { Total vertical pressure = V ;
 " horizontal " = H ;

$$V = \pi r'^2 p \quad \text{.......} \quad (1)$$

$H : V :: \sqrt{r^2 - r'^2} : r' \quad \therefore H = \frac{V}{r'} \sqrt{r^2 - r'^2} \quad \therefore H = \pi r' p \sqrt{r^2 - r'^2}$

Calling the intensity of the horizontal pressure p_h, and the length, or thickness, of the flange in the direction of the axis of the cylinder l, [see Fig. 1] we have.

$$p_h = \frac{H}{2\pi r' l} = \frac{\pi r' p}{2\pi r' l}\sqrt{r^2 - r'^2} = \frac{p}{2l}\sqrt{r^2 - r'^2} \quad \therefore$$

$$\text{Thrust} = p_h \cdot r' l = \tfrac{1}{2} p r' \sqrt{r^2 - r'^2} \quad \text{.......} \quad (2.)$$

If $r = r'$, as in Fig. 1, then thrust = 0;
" $r = \infty$, or the end is a plane, the thrust will be infinite.

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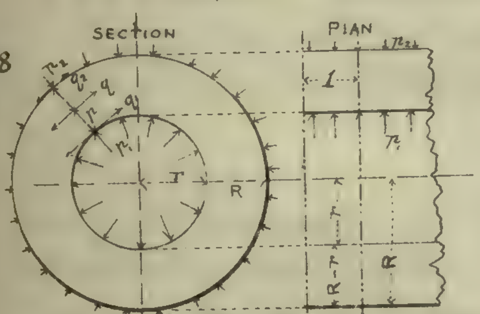

Fig. 1

THICK HOLLOW CYLINDER. [see A.M. Art. 273]

The assumption that the circumferential tension, or hoop-tension as it may be called, in a hollow cylinder is uniformly distributed, is approximately true only when the thickness is small as compared with the radius; for if a ring of the cylinder be conceived to be divided into several concentric hoops, one within another the tension of the innermost hoop balances part of the radial pressure of the confined fluid, so that a diminished radial pressure is transmitted to the second hoop which has therefore

p.228 a less tension than the first hoop, and so on.

§152 Eq.-(1) § 150, [p.59 Notes], replacing f by q to make it general, is $p = \frac{qt}{r}$ ∴ $q = \frac{pr}{t}$, which gives the *mean* hoop-tension in a thick as well as in a thin cylinder; but it is not the mean, but the greatest hoop-tension [that is tension round the inner surface of the cylinder], which is limited by the strength of the material. The object of the present investigation is to show what law the variation of hoop tension follows, and thence, what relation the maximum tension bears to the fluid pressure.

Cont.

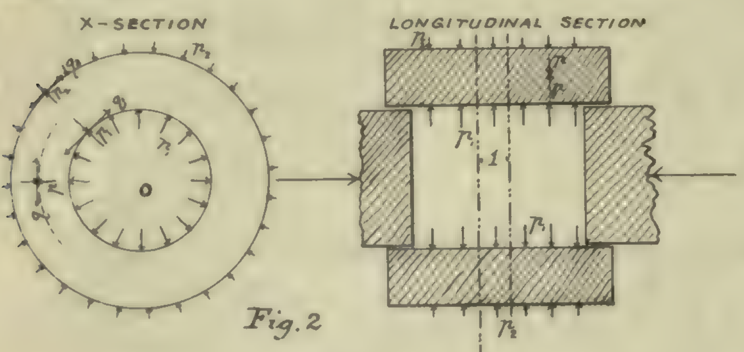

Fig. 2

To make the solution perfectly general, it will be supposed that the cylinder is pressed from without as well as from within. Let Fig. 2 represent the cylinder; R = external radius; r = internal " ;

p_1 = fluid pressure from within; q_1 = hoop-tension at inner surface
p_2 = " " " without; q_2 = " " " outer "

Consider a ring whose length, parallel to the axis of the cylinder, is unity. The radial section of that ring, from r to R, has to sustain the difference between the total pressures from within and without, in a direction perpendicular to the radius, on a quadrant bounded by that radius. That difference is

$$p_1 r - p_2 R.$$

Conceive the ring to be divided into an indefinite number of concentric hoops, each of the thickness dr, and exerting a tension of the intensity q; then the total hoop-tension will be

$$\int_r^R q\,dr = p_1 r - p_2 R \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

From the symetry of the ring and of the forces acting on it in all directions around the centre O, it is obvious that the axes of stress of any particle of metal must be respectively in the direction of a radius, and perpendicular to that direction. The <u>principal</u> <u>stresses</u> at any particle are a radial pressure, p [which for each particle at the inner surface is p_1 and for each particle at the outer surface is p_2] and a hoop-tension q.

228 As in the case of the ellipse of stress, p. 169, [Notes p. 38]
152 we may conceive this pair of principal stresses to be made up
Cont. of two component pairs, viz:—

A pair of equal stresses of the same kind, constituting a fluid <u>pressure</u> or <u>tension</u>, whose common intensity, stated so as to be a tension when positive, a pressure when negative is

$$\frac{q-p}{2} = m;$$

and a pair of stresses of contrary kinds, whose common intensity is

$$\frac{q+p}{2} = n.$$

Thus we have $q = n + m$; $p = n - m$, and the problem is to be solved by first supposing m to act alone, then supposing n to act alone, and lastly combining their effects; observing, that the only solutions of equation (a) which are admissible, are those which are true for all values of R and r.

CASE I *Equal and similar stresses*, or $n = 0$. In this case

$$q = -p = m$$

showing that instead of a radial pressure, there is a radial tension equal to the hoop-tension and constituting along with it simply a fluid tension [Notes p. 40] of the intensity m at each point. Equation (a) is fulfilled by making

$$q = -p = m = \text{constant}, \quad \text{.................} \quad (b)$$

which reduces both sides of equation (a) to

$$m(R - r).$$

CASE II *Equal and contrary stresses*, or $m = 0$. In this case

$$q = p = n$$

and the solution of equation (a) $\left\{ \int_r^R q\,dr = p_1 r - p_2 R \right\}$ is

$$q = p = n = \frac{a}{r'^2} \quad \text{.........................} \quad (c)$$

a being an arbitrary constant, and r' any value of the radius, from r to R inclusive; for this reduces both sides of equation (a) to

$$a\left(\frac{1}{r} - \frac{1}{R}\right).$$

CASE III *General solution*. By combining the two partial solutions, equations (b) and (c) together, we find

$$\left.\begin{aligned} &\text{Radial pressure}, \quad p = n - m = \frac{a}{r'^2} - m; \\ &\text{Hoop-tension}, \quad q = n + m = \frac{a}{r'^2} + m; \end{aligned}\right\} \quad \text{..................} \quad (d)$$

To determine the constants, a and m, we have the equations

$$\frac{a}{r^2} - m = p_1; \quad \frac{a}{R^2} - m = p_2; \quad \text{.................} \quad (e)$$

subtracting the second from the first.

p.228 $$\frac{a}{r^2} - \frac{a}{R^2} = p_1 - p_2 \therefore a = \frac{(p_1 - p_2) R^2 r^2}{R^2 - r^2}$$

§152 Ext. Also from eqs. (e) $a - r^2 m = p_1 r^2$ and $a - R^2 m = p_2 R^2 \therefore m = \frac{p_1 r^2 - p_2 R^2}{R^2 - r^2}$ (f.)

giving for the final hoop-tension at the inner surface where it is a maximum

$$q_1 = \frac{a}{r^2} + m = \frac{p_1(R^2+r^2) - 2p_2 R^2}{R^2 - r^2} \qquad \text{(g.)}$$

We may write $p_1 - p_2 = p$, where p is the excess of the inside pressure over the outside, and is the same as used in the text.

$$q_1 = \frac{p(R^2+r^2) + p_2(R^2+r^2) - 2p_2 R^2}{R^2 - r^2};$$

if p_2 is small compared with p the terms $p_2(R^2+r^2) - 2p_2 R^2 = p_2(r^2 - R^2)$ may be discarded which leaves $q_1 = \frac{p(R^2+r^2)}{R^2 - r^2}$ ......(h.)

Replacing q_1 by f we have $$\frac{p}{f} = \frac{R^2 - r^2}{R^2 + r^2} \qquad \text{(1.)}$$

$$\therefore \begin{cases} R^2 - r^2 = cp \\ R^2 + r^2 = cf \end{cases} \quad \therefore \left.\begin{array}{l} R^2 = \frac{c}{2}(p+f) \\ r^2 = \frac{c}{2}(f-p) \end{array}\right\} \therefore \frac{R}{r} = \sqrt{\frac{f+p}{f-p}} \qquad \text{(2.)}$$

If $p \geqq f$, we have $\frac{R}{r} \geqq \frac{\sqrt{f+p}}{0} \geqq \infty$, from which we see that no thickness however great will hold the fluid.

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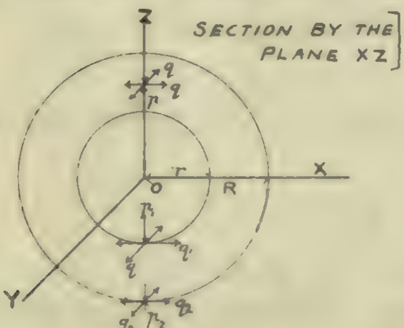
p.229 THICK HOLLOW SPHERE. [A.M. 275]

§153



The figure represents a diametral section of a hollow sphere, the fluid pressure within and without being p_1 and p_2 as before. The pressure to be resisted at the section is

$$\pi(p_1 r^2 - p_2 R^2);$$

and if the section of the metal be conceived to be divided into an indefinite number of concentric rings, the breadth of one of these rings being dr, its radius r', and the tension at it q, it appears that the total resistance of the section will be

$$2\pi \int_r^R q r' dr;$$

and hence the equation to be fulfilled for all values of p_1, p_2, r and R is

$$2\int_r^R q r' dr = p_1 r^2 - p_2 R^2 \qquad \text{(a)}$$

From symetry it appears that the axes of stress at any particle must be, one in the direction of a radius, with the pressure p along it, and the other two in any directions

229 perpendicular to the first and to each other, with equal tensions
153 q along them. Two partial solutions are obtained in the following manner: –

Let
$$\frac{2q-p}{3} = m, \quad \frac{q+p}{3} = n;$$
so that
$$q = n + m; \quad p = 2n - m.$$

CASE I. $n = 0$, $q = -p = m$; being the case of *fluid tension*, equal in all directions [this has been proven for forces parallel to one plane, Notes p. 40, and, since this plane may be taken in any direction, it is generally true]. In this case, equation (a) is solved by making
$$q = -p = m = \text{constant}, \qquad \text{(b)}$$
which reduces both sides of that equation to
$$m(R^2 - r^2).$$

CASE II $m = 0$, $q = \frac{p}{2} = n$; being the case of a pair of circumferential tensions each equal to half the radial pressure. In this case, equation (a) is solved by making
$$q = \frac{p}{2} = n = \frac{a}{r'^3}; \qquad \text{(c)}$$
which substituted in that equation gives
$$\left\{2\int_r^R p\, r'dr = p_1 r - p_2 R\right\} \therefore \ 2\int \frac{a}{r'^3}\cdot r'dr = \frac{a r^2}{r^3} - \frac{a R^2}{R^3},$$
both sides reducing to
$$2a\left(\frac{1}{r} - \frac{1}{R}\right).$$

CASE III General solution.
$$\left.\begin{aligned} p &= 2n - m = \frac{2a}{r'^3} - m \\ q &= n + m = \frac{a}{r'^3} + m \end{aligned}\right\} \qquad \text{(d)}$$
The constants, a and m, are deduced from the equations
$$p_1 = \frac{2a}{r^3} - m; \quad p_2 = \frac{2a}{R^3} - m$$
$$\left.\begin{aligned} &\therefore p_1 - p_2 = \frac{2a}{r^3} - \frac{2a}{R^3} \quad \cdots\cdots \quad a = \frac{(p_1 - p_2)R^3 r^3}{2(R^3 - r^3)} \\ &\left.\begin{aligned} &\text{also } p_1 r^3 = 2a - m r^3 \text{ and } p_2 R^3 = 2a - m R^3 \\ &\therefore m(R^3 - r^3) = p_1 r^3 - p_2 R^3 \cdots\cdots \end{aligned}\right\} \therefore m = \frac{p_1 r^3 - p_2 R^3}{R^3 - r^3}; \end{aligned}\right\} \qquad \text{(e)}$$
giving finally, for the maximum tension,
$$q_1 = \frac{a}{r^3} + m = \frac{p_1(R^3 + 2r^3) - 3p_2 R^3}{2(R^3 - r^3)} \qquad \text{(f)}$$
Making as before $p_1 - p_2 = p$ and supposing p_2 small compared with p_1, or p, we may drop the term $p_2(R^3 + 2r^3) - 3p_2R^3$, which gives
$$q_1 = \frac{p(R^3 + 2r^3)}{2(R^3 - r^3)}.$$

p.229 §153 Cont. Replacing q by f we have

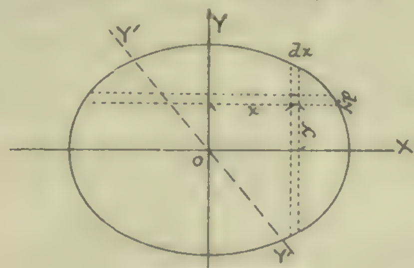
$$\frac{p}{f} = \frac{2R^3 - 2r^3}{R^3 + 2r^3} \qquad \text{(1.)}$$

$$\therefore \left.\begin{aligned} 2R^3 - 2r^3 &= cp \\ R^3 + 2r^3 &= cf \end{aligned}\right\} \therefore \quad 2R^3 + 4r^3 = 2cf \quad \therefore \quad \left.\begin{aligned} R^3 &= \frac{c(f \pm p)}{3} \\ r^3 &= \frac{c(2f - p)}{6} \end{aligned}\right\} \therefore$$

$$\frac{R}{r} = \sqrt[3]{\frac{2f + 2p}{2f - p}} \qquad \text{(2.)}$$

§154 *N.B. A full discussion is beyond the scope of these notes. See A.M. Arts. 249 - 264, pp. 275-28*

p.232 §157 MOMENT OF INERTIA



Let OX and OY be two axes passing through the centre of gravity and at right angles to each other, dividing the figure symetrically, then the expressions

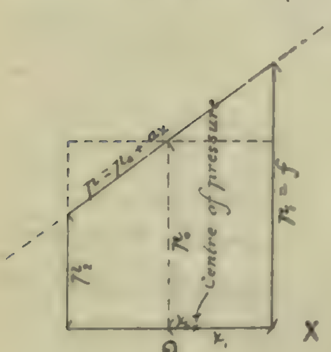
$$\left.\begin{aligned} \iint x^2\,dx\,dy &= \int x^2 y\,dx = I \\ \iint y^2\,dx\,dy &= \int y^2 x\,dy = J \end{aligned}\right\} \text{are called the principal}$$

moments of inertia of the surface; a term borrowed from dynamics. I and J are the maximum and minimum moments of inertia respectively.

These quantities enter into many discussions in engineering as, for example, in discussing uniformly varying stress; see Eq.(7) p. 163, §106.

The moment of inertia may be taken about any axis as OY' but, as in most cases occurring in practice, the maximum and minimum moments only are required the others are not discussed here; the reader is referred to [A.M. § 95 pp. 77 - 82].

In this article [157] the deviation of the centre of pressure from the centre of figure is along one of the principal axes; and, unless this is the case, equation (2) is not strictly true.



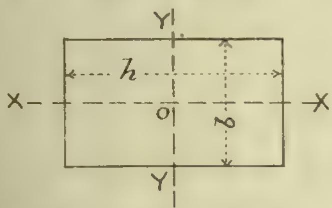
$f = p_0 + ax_1$ and from Eq(7) p.163 $x_0 = \dfrac{aI}{P}$ where $I = \iint x^2\,dx\,dy$ and $P = p_0 S$; $\therefore\ a = \dfrac{x_0 p_0 S}{I}$;

$$\therefore \frac{p_0}{f} = \frac{p_0}{p_0 + x_0 p_0 \frac{S}{I} x_1} = \frac{1}{1 + \frac{x_0 x_1 S}{I}} = \frac{p_0 S}{fS} = \frac{P}{fS};$$

$$\therefore P = \frac{fS}{1 + \frac{x_0 x_1 S}{I}} \qquad \text{(2.)}$$

232 The values of I, J and $\frac{x_1 S}{I}$, for the cross-sections which most
157 frequently occur in practice are found as follows.

I RECTANGLE.



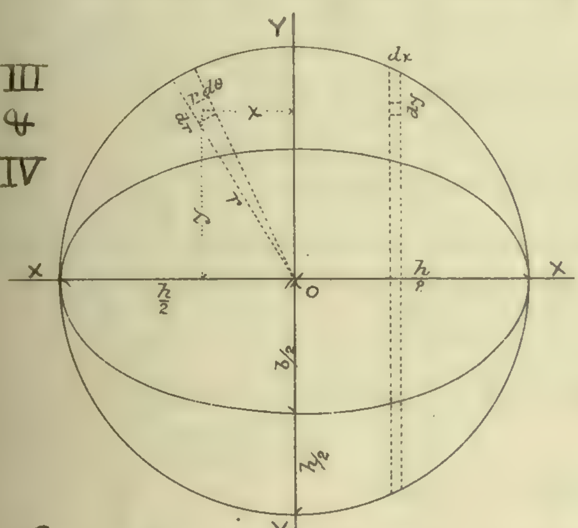
$$I = \int x^2 y\, dx = b\int_{-\frac{h}{2}}^{+\frac{h}{2}} x^2\, dx = \tfrac{1}{3}\left(\tfrac{h}{2}\right)^3 b + \tfrac{1}{3}\left(\tfrac{h}{2}\right)^3 b$$

$$\therefore I = \frac{h^3 b}{12}\ ;\ \text{similarly}\ J = \frac{h b^3}{12} \quad \text{.............(a.)}$$

$$\frac{x_1 S}{I} = \frac{\frac{1}{2}h \cdot hb}{\frac{1}{12}h^3 b} = \frac{6}{h} \quad \text{.............(a'.)}$$

II SQUARE. In this case $h = b$ $\therefore I = J = \frac{h^4}{12}$ and $\frac{x_1 S}{I} = \frac{6}{h}$ ....(b.)

III & IV



ELLIPSE. $I = \int x^2 y\, dx = \int x^2 \left(2\frac{b}{h}\sqrt{\frac{h^2}{4} - x^2}\right) dx$

$= 2\frac{b}{h}\int \sqrt{\frac{h^2}{4} - x^2} \cdot x^2\, dx$.

CIRCLE. $I = \int x^2 y\, dx = 2\int \sqrt{\frac{h^2}{4} - x^2} \cdot x^2\, dx$.

Thus we see that the I of the ellipse may be obtained from the I of the circle by multiplying by $\frac{b}{h}$ i.e.

$$I_e = I_c \frac{b}{h}.$$

For the circle $I = J$ $\therefore I + J = 2I = \iint (x^2 + y^2)\, dx\, dy$ ....(c.)
Passing to polar coördinates $x^2 + y^2 = r^2$ and the differential of the surface corresponding to $\underline{dx\, dy}$ is $\underline{r\, dr \cdot d\theta}$.

$$\therefore 2I = \int_0^{\frac{h}{2}} r^3 dr \int_0^{2\pi} d\theta = 2\pi \int_0^{\frac{h}{2}} r^3 dr = \frac{2\pi}{4}\left(\frac{h}{2}\right)^4 = \frac{\pi h^4}{32}\ \therefore$$

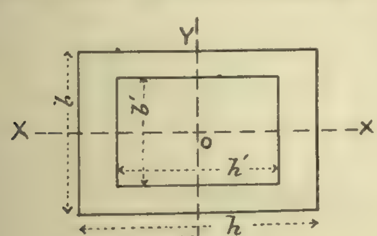
CIRCLE.

$$\begin{cases} I = \dfrac{\pi h^4}{64}\ ;\ \text{similarly}\ J = \dfrac{\pi b^4}{64} & \text{.............(d.)} \\[2ex] \dfrac{x_1 S}{I} = \dfrac{\frac{h}{2}\,\pi\left(\frac{h}{2}\right)^2}{\frac{\pi h^4}{64}} = \dfrac{8}{h} & \text{.............(d'.)} \end{cases}$$

ELLIPSE.

$$\begin{cases} I = \dfrac{\pi h^4}{64} \cdot \dfrac{b}{h} = \dfrac{\pi h^3 b}{64}\ ;\ \text{similarly}\ J = \dfrac{\pi h b^3}{64}; & \text{.............(e.)} \\[2ex] \dfrac{x_1 S}{I} = \dfrac{\frac{h}{2} \cdot \pi \cdot \frac{h}{2} \cdot \frac{b}{2}}{\frac{\pi h^3 b}{64}} = \dfrac{8}{h} & \text{.............(e.)} \end{cases}$$

V HOLLOW RECTANGLE.



$$I = \tfrac{1}{12}\left[h^3 b - h'^3 b'\right]\ ;\ J = \tfrac{1}{12}\left[h b^3 - h' b'^3\right] \quad \text{.............(f.)}$$

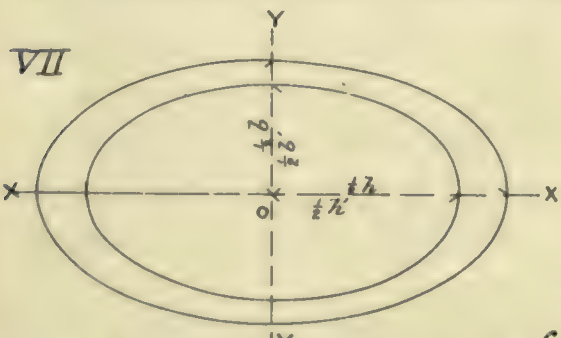
$$\frac{x_1 S}{I} = \frac{\frac{h}{2}(hb - h'b')}{\frac{1}{12}(h^3 b - h'^3 b')} = \frac{6h(hb - h'b')}{h^3 b - h'^3 b'} \quad \text{.............(f'.)}$$

Moments of Inertia & $\frac{x_1 S}{I}$

p.234
§157
Cont.

Figure of Cross-section	Maximum I neutral axis OY.	Minimum J neutral axis OX.	$\frac{x_1 S}{I}$
I. Rectangle. Length along OX, h; breadth " OY, b.	$\frac{h^3 b}{12}$	$\frac{h b^3}{12}$	$\frac{6}{h}$
II. Square. — Side $= h$.	$\frac{h^4}{12}$	$\frac{h^4}{12}$	$\frac{6}{h}$
III. Ellipse. — Longer axis h. Shorter " b.	$\frac{\pi h^3 b}{64}$	$\frac{\pi h b^3}{64}$	$\frac{8}{h}$
IV. Circle. — Diameter h. $\left[\frac{\pi}{64} = \frac{1}{20.4} \text{ nearly}\right]$	$\frac{\pi h^4}{64}$	$\frac{\pi h^4}{64}$	$\frac{8}{h}$
V. Hollow Rectangle. — Outside h, b. Inside h', b'.	$\frac{1}{12}(h^3 b - h'^3 b')$	$\frac{1}{12}(h b^3 - h' b'^3)$	$\frac{6h(hb - h'b')}{h^3 b - h'^3 b'}$
VI. Hollow Square. — Outside h. Inside h'.	$\frac{1}{12}(h^4 - h'^4)$	$\frac{1}{12}(h^4 - h'^4)$	$\frac{6h}{h^4 - h'^4}$
VII. Hollow Ellipse. — Outside diam's. $h. b.$ Inside " $h' b'$.	$\frac{\pi}{64}(h^3 b - h'^3 b')$	$\frac{\pi}{64}(h b^3 - h' b'^3)$	$\frac{8h(hb - h'b')}{h^3 b - h'^3 b'}$
VIII. Circular Ring. — Outside diam h. Inside " h'.	$\frac{\pi}{64}(h^4 - h'^4)$	$\frac{\pi}{64}(h^2 - h'^2)$	$\frac{8h}{h^2 + h'^2}$
IX. Isosceles Triangle; base, b; height, h; x_1 measured from summit.	$\frac{h^3 b}{36}$		$\frac{12}{h}$
X. Symmetrical Assemblage of Rectangles; dimensions of any one $h \parallel x$, $b \parallel y$; distance of its centre from OY, x_0; from OX, y_0.	$\Sigma \cdot \frac{h^3 b}{12} + \Sigma \cdot h b x_0^2$	$\Sigma \cdot \frac{h b^3}{12} + \Sigma \cdot h b y_0^2$	

VII

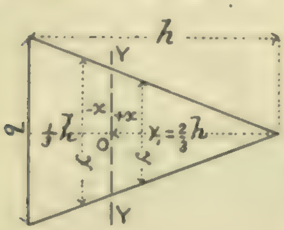


HOLLOW ELLIPSE

$$I = \frac{\pi}{64}(h^3 b - h'^3 b') \; ; \quad J = \frac{\pi}{64}(h b^3 - h' b'^3)$$

$$\frac{x_1 S}{I} = \frac{\frac{h}{2}\left(\pi \frac{hb}{4} - \pi \frac{h'b'}{4}\right)}{\frac{\pi}{64}(h^3 b - h'^3 b')} = \frac{8h(hb - h'b')}{h^3 b - h'^3 b'}.$$

IX Isosceles Triangle. $I = \iint x^2\,dx\,dy = \int x^2 y\,dx$; but $\left\{ \begin{array}{l} y : b :: \frac{2}{3}h - x : h \\ \therefore y = \frac{b}{h}\left(\frac{2}{3}h - x\right) \end{array} \right.$



$$\therefore I = \int \frac{b}{h}\left(\tfrac{2}{3}h - x\right) x^2\,dx$$

$$= b\left[\frac{2}{3}\cdot\frac{1}{3}x^3 - \frac{1}{h}\cdot\frac{1}{4}x^4\right] + C;$$

between the limits $-\frac{1}{3}h$ and $+\frac{2}{3}h$,

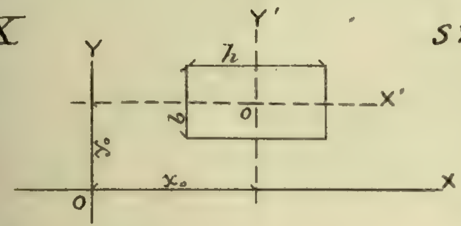
$$I = b\left[\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}h^3 + \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}h^3 + \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}h^3 - \frac{1}{4}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}h^3\right],$$

.234 $\therefore\ I = b\left[\frac{18}{3\cdot3\cdot3\cdot3}h^3 - \frac{15}{4\cdot3\cdot3\cdot3\cdot3}h^3\right] = \frac{h^3 b}{36}$

157 $\frac{x_1 S}{I} = \frac{\frac{2}{3}h\cdot\frac{1}{2}hb}{\frac{h^3 b}{36}} = \frac{12}{h}$.

art.

X



SYMMETRICAL ASSEMBLAGE OF RECTANGLES.

To find the moment of inertia of any surface about an axis which does not pass through the centre of gravity.

Let OX′ and OY′ pass through the centre of gravity. $x = x_0 + x'$; $y = y_0 + y'$; $dx = dx'$; $dy = dy'$.

$$I = \iint x^2\,dx\,dy = \iint (x_0^2 - 2x_0x' + x'^2)\,dx\,dy$$

$$= x_0\iint dx'dy' - 2x_0\iint x'\,dx'dy' + \iint x'^2\,dx'\,dy'$$

Since OX and OY pass through the centre of gravity $\iint x'\,dx'dy' = 0$.

$$\therefore\ I = x_0^2 S + I' \text{ ; also } J = y_0^2 S + J'$$

Symmetrical assemblage of rectangles; dimensions of any one $h, \parallel x$; $b, \parallel y$;

$$I = \Sigma(hb\,x_0^2) + \Sigma\frac{h^3 b}{12} \text{ ; } J = \Sigma(hb\,y_0^2) + \Sigma\cdot\frac{hb^3}{12}\ldots$$

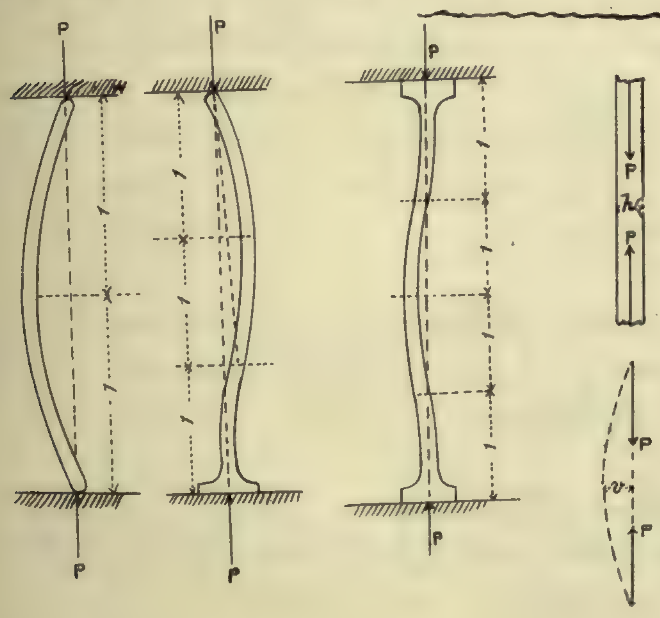
p.235 II CRUSHING BY SHEARING OR SLIDING. The shearing or tangential stress on any plane [p.22, Notes] is $p_t = p\sin\theta\cos\theta$.

For a max. $\frac{dp_t}{d\theta} = p(-\sin^2\theta + \cos^2\theta) = 0$ $\therefore \sin\theta = \cos\theta$ $\therefore \tan\theta = 1$ $\therefore \theta = \text{arc}\tan 1$

or $\theta = 45°$

p.237

§.158



p' = intensity due to direct pressure

p'' = " of pressure due to bending moment.

$$p' = \frac{P}{S}$$

$\left\{\begin{array}{l}\text{C.E. Eq. 5 p.252, § 162.}\\ M_0 = nfbh^2 \ \therefore f = \frac{M_0}{nbh^2}\end{array}\right\}$

$M_0 = Pv$ $\therefore p'' \propto \frac{Pv}{bh^2}$

$\left\{\text{from eq. 13 p. 273, § 169. } v \propto \frac{l^2}{h}\right\}\therefore$

$$p'' \propto \frac{Pl^2}{bh^3} \propto \frac{Pl^2}{bh\cdot h^2} \propto \frac{Pl^2}{Sh^2}\ \therefore$$

p.237 §158 $p'' = a\frac{P}{S}\cdot\frac{l^2}{h^2}$ where a is some constant to be determined by experiment.

$$f = p' + p'' = \frac{P}{S} + a\frac{P}{S}\frac{l^2}{h^2} = \frac{P}{S}\left(1 + a\frac{l^2}{h^2}\right)$$

$$\therefore P = \frac{fS}{1 + a\frac{l^2}{h^2}} \quad \text{.........} (4).$$

For pillars rounded or jointed at both ends substitute $2l$ for l, see sketch on last page. $$P = \frac{fS}{1 + 4a\frac{l^2}{h^2}} \quad \text{.........} (5.)$$

The following values of the constants a and f are given by Prof. Allan as having been obtained by taking the average of those given by several of the best authors.

	f	a
For wood - rect. solid pillars	6,000	1/350
" " round " "	6,000	4/3(1/350)
For cast iron - solid round pillars	80,000	1/325
" " " " rect. "	8.0,000	3/4 (1/325)
" " " - hollow cylindrical either square or circular section	80,000	1/500
For wrought iron - solid rectangular pillars ...	36,000	1/3000
" " " " round " ...	36,000	4/3 (1/3000)
" " " hollow thick cylinders ...	36,000	1/5,000

But wrought iron hollow columns usually buckle. Against that the strength of well shaped and well made columns is 27,000 lbs. pr. sq. in. of section.

For steel (mild) solid round pillars	67,000	1/1400
" " " " rect. "	67,000	3/4 (1/1400)
" " " hollow cylinders	67,000	1/2500
For steel (strong) - solid round pillars	114,000	1/900
" " " " rect. "	114,000	3/4 (1/900)
" " " hollow cylinders	114,000	1/1500

For pillars with both ends flat the above values apply.
" " " " rounded take $\frac{1}{3}$ of the above values.
" " " one end rounded " 2/3 " " " "

p.238 When the centre of pressure deviates from the axis of the
§158 strut we use eq.(2) § 157, p. 233

$\left\{ P = \dfrac{fS}{1+\frac{x_0 x_1 S}{I}} \right\}$; calling this f, f_1 for destinction, we have,

$$f_1 = \frac{P}{S}\left(1 + x_0 x_1 \frac{S}{I}\right).$$

This f_1 is composed of two parts $p' = \frac{P}{S}$ and $p''' = \frac{P}{S} x_0 x_1 \frac{S}{I}$ but $x_0 = x$, $x_1 = \frac{h}{2}$, and $\frac{S}{I} = \frac{1}{r^2}$; where r is the "radius of gyration".

$$\therefore f_1 = p' + p''' = \frac{P}{S}\left(1 + \frac{xh}{2r^2}\right)$$

$$\therefore f = p' + p'' + p''' = \frac{P}{S}\left(1 + a\frac{l^2}{h^2} + \frac{xh}{2r^2}\right) \therefore P = \frac{fS}{1 + a\frac{l^2}{h^2} + \frac{xh}{2r^2}} \quad \ldots\ldots\ldots (6)$$

The values of $\frac{xh}{2r^2} = \frac{xh}{2}\cdot\frac{S}{I} = x\,\frac{\frac{h}{2}S}{I}$ are easily obtained from the table on p. 234, since $\frac{h}{2}$ is the same as x_1, by multiplying by x.

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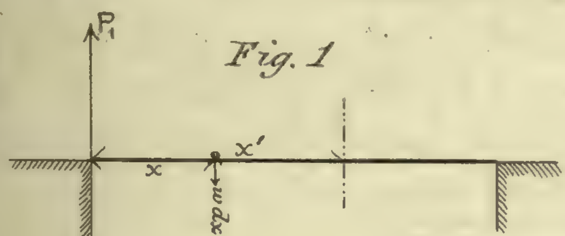
p.243
§160

Fig. 1



Eq.(10) $M = \int_0^x F\,dx \quad \therefore \frac{dM}{dx} = F$

Substituting in eq.(10) for F its value from eq's. (6),(7)

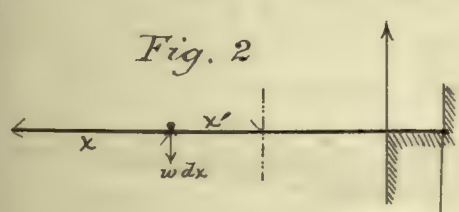
$$M = \int_0^{x'} P_1\,dx - \int_0^{x'}\int_x^{x'} w\,dx^2 .$$

The limits, as may be seen from the figure, are the correct ones, instead of writing as our author does $\int_0^{x'}\int_0^{x} w\,dx^2$, for as the force P_1 must be multiplied by each dx from its point of application, o, to the plane of section x'; so also the force $w\,dx$ must be multiplied by each dx from its point of application, x, to the plane of section x'.

$$\therefore M = P_1 x' - \int_0^{x'}(x' - x)\,w\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (11.)$$

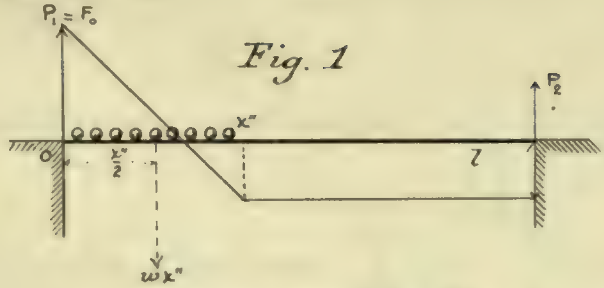
This equation is evidently true since each force is multiplied by its lever arm.

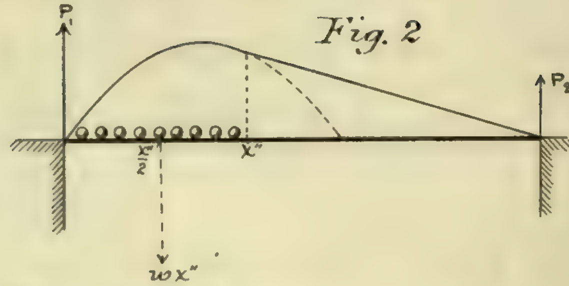
Fig. 2



$$M = \int_0^{x'}\int_x^{x'} w\,dx^2 = \int_0^{x'}(x' - x)\,w\,dx \quad \ldots\ldots\ldots (12)$$

p. 246 § 161 CASE VII.





$$P_1 = \frac{wx''(l-\frac{x''}{2})}{l}\ ;\ P_2 = \frac{wx''^2}{2l} \quad \text{Therefore}$$

$$F = P_1 - \int w\,dx' = \frac{wx''}{l}(l-\tfrac{x''}{2}) - wx' \quad \text{— between } x'' \text{ and origen —}$$
$$= -\frac{wx''^2}{2l} \quad \text{— beyond } x'' \text{ —}$$

$$M = \int F\,dx = \frac{wx''}{l}(l-\tfrac{x''}{2})x' - \frac{wx'^2}{2} \quad \text{---- between } x'' \text{ and origen,}$$
$$= \frac{wx''^2}{2l}(l-x') \quad \text{----- beyond } x'' \text{ —}$$

M is a maximum when $\frac{dM}{dx} = 0 = F$; that is where the shearing force reduces to 0.

Fig. 1 graphically represents the shearing force, Fig. 2 the moments; each being represented by the ordinate. In the latter case the curve consists of a parabola and a straight line tangent to it at x″.

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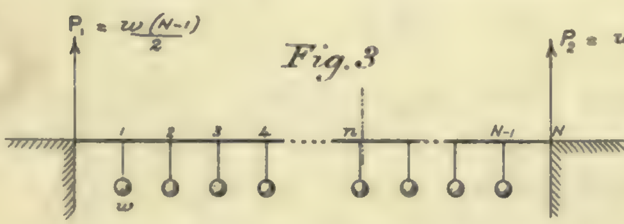
p. 247 § 161 Cont. CASE VIII.



The value of F is easily found. $\{M = \Delta x\ \Sigma F\}$ eq. (9) § 160, p. 243.

$F = w\left[\frac{N-1}{2} - n\right]$, which is the general expression.

$F_0 = w\left[\frac{N-1}{2}\right]$; $F_1 = w\left[\frac{N-1}{2} - 1\right]$; $F_2 = w\left[\frac{N-1}{2} - 2\right]$; $F_3 = w\left[\frac{N-1}{2} - 3\right]$ &c.

Hence the successive values of F constitute a descending arithmetical series whose first term $= w(\frac{N-1}{2})$, and whose last term is $= w\left(\frac{N-1}{2} - n\right]$. In finding M, for any section we use n terms of this series and the n^{th} term is $= w\left[\frac{N-1}{2} - (n-1)\right]$. Common difference, -1.

The formula for sum of arithmetical series is

$s = \frac{1}{2}n'(2a + (n'-1)d)$. In this case $S = \frac{1}{2}n(N-1-n+1)w = \frac{1}{2}wn(N-n)$.

p 247 $\Delta x = \frac{l}{N} \quad \therefore \quad M = \frac{l}{N} \cdot \frac{1}{2} wn \left[N - n\right] = \frac{n(N-n)\,wl}{2N}$

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Cont. To find m. When N is even, $n = \frac{N}{2}$;

" N " odd, $n = \frac{N-1}{2}$, or $= \frac{N+1}{2}$.

$$\therefore m = \frac{M}{Wl} \text{ (in the 1}^{\text{st.}}\text{ case)} = \frac{\frac{N}{2}\left(N - \frac{N}{2}\right)wl}{2N} \div w(N-1)\,l = \frac{\frac{N}{8}}{N-1}$$

$$\text{(in the 2}^{\text{nd.}}\text{ case)} = \frac{\frac{N-1}{2}\left(N - \frac{N-1}{2}\right)wl}{2N} \div w(N-1)\,l = \frac{N^2-1}{8N(N-1)} = \frac{N+1}{8N}$$

Rankine obtains his 1st. result by making the total load $W = Nw$ instead of $(N-1)w$. By using this value we find in the first case,

$$m = \frac{\frac{\frac{N}{2}\left(N - \frac{N}{2}\right)wl}{2N}}{Nwl} = \frac{1}{8}.$$

CASE IX.

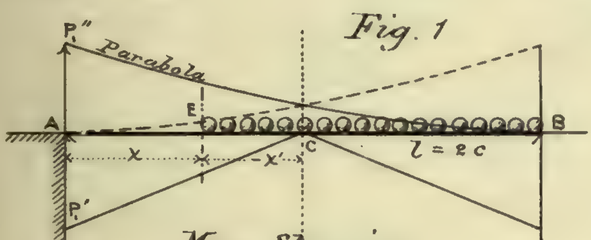


Fig. 1

Max. Shearing Force

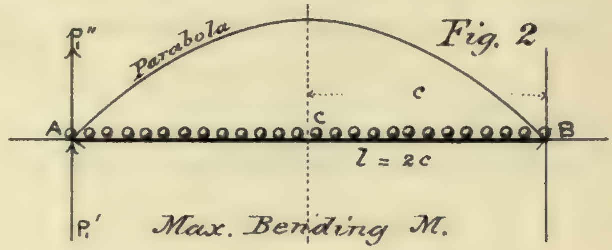


Fig. 2

Max. Bending M.

$$\left.\begin{array}{l} P_1' \text{ refers to the permanent load } w \\ P_1'' \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{moving} \quad \text{"} \quad w' \end{array}\right\} \therefore \; P_1 = P_1' + P_1''$$

$$P_1' = w\frac{l}{2}\,; \quad P_1'' : w'(l-x) :: l-x : l \quad \therefore P_1'' = \frac{w'(l-x)^2}{l}$$

$$F = P_1' - \int w\,dx + P_1'' = w\left(\frac{l}{2} - x\right) + \frac{w'(l-x)^2}{l} \quad \text{.......... (2.)}$$

For the origen at the centre and $+x'$ measured in the same direction, that is to the right, we have

$$x = c + x'\,; \qquad F = -wx' + \frac{w'(c-x')^2}{2c}.$$

For a beam loaded as above, on the end next B, x' is negative for maximum values. Replacing x' in the above equation by $-x'$ we get

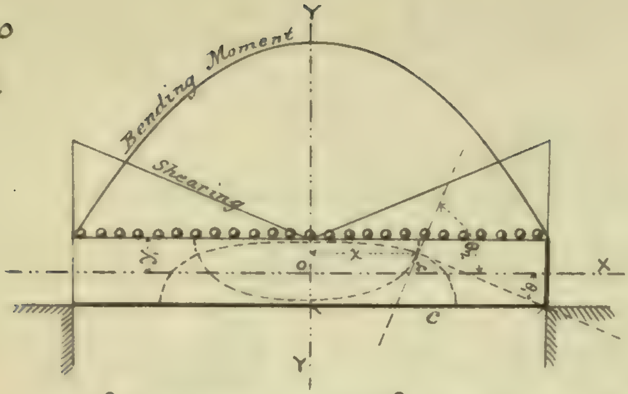
$$F = wx' + \frac{w'(c+x')^2}{2c} \quad \text{.......... (3.)}$$

$$M = \frac{(w+w')(l-x)\,x}{2} \quad \text{and} \quad x = c + x' \quad \therefore \quad M = \frac{(w+w')(2c-c-x')(c+x')}{2}$$

$$\therefore M = \frac{(w+w')(c^2-x'^2)}{2} \quad \text{.......... (4)}$$

which is the equation of a parabola.

p. 250
§162



DIRECTION OF STRESS.

$\left\{\tan 2\theta = \frac{2q}{p_x - p_y}\right\}$ § 108, eq. (4) p. 170

In this equation q is the intensity of the shearing force at any point in a direction perp. to OX; p_x is the intensity of stress, due to the bending moment, in the direction of OX. p_y is the intensity of stress perpendicular to OX and is therefore $=0$. Replacing p_x by p we have

$$\tan 2\theta = \frac{2q}{p} \quad \text{------------(a)}$$

When $p = 0$, $\tan 2\theta = \infty$ $\therefore$ $\theta = 45°$ or $135°$
" $q = 0$, $\tan 2\theta = 0$ $\therefore$ $\theta = 0$ " $90°$

From p. 41 Eq. (12) $\tan 2\theta = \frac{2}{\cot\theta - \tan\theta} = \frac{2}{\frac{1-\tan^2\theta}{\tan\theta}}$

$\therefore \frac{\tan 2\theta}{\tan\theta} = \frac{2}{1-\tan^2\theta}$ $\therefore \frac{\tan\theta}{\tan 2\theta} = \frac{1-\tan^2\theta}{2}$ $\therefore \tan^2\theta + \frac{2\tan\theta}{\tan 2\theta} = 1$

$\therefore \tan\theta = -\frac{1}{\tan 2\theta} \mp \sqrt{1 + \frac{1}{\tan^2 2\theta}}$; or $-\tan\theta_1 = \frac{1}{\tan\theta_2} = \frac{p}{2q} \pm \sqrt{1 + \frac{p^2}{4q^2}} = -\frac{dy}{dx}$ ...(b)

From eq. (3) § 108, p. 170 the above $= \frac{p_1}{q}$, where p_1 is the new principal stress.

Equation (b) is the differential equation of all the curves; the constants of integration, will give any particular curve.

p and q may each be expressed in terms, of x and y, but the actual integration is difficult. From eq. (d), below, we have

$$M = \frac{pI}{y} \therefore p = \frac{My}{I}.$$

For an uniformly loaded rectangular beam $M = \frac{w}{2}(c^2 - x^2)$ and $I = \frac{h^3 b}{12}$ $\therefore$

$$p = \frac{6w(c^2 - x^2)}{h^3 b} y.$$

From § 168, p. 266 we have [Notes p. 81, eq. (c)]

$$q = \frac{F}{I}\int_y^{y_1} y\,dy = \frac{F}{2I}(y_1^2 - y^2)$$

but $F \propto x$ or $F : wc :: x : c$; $\therefore F = wx$;

$$q = (y_1^2 - y^2)\frac{6wx}{h^3 b} \quad \therefore \frac{p}{2q} = \frac{(c^2 - x^2)y}{x(y_1^2 - y^2)} ;$$

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.162 $\therefore \quad -\frac{dy}{dx} = \frac{(c^2-x^2)y}{2x(y_1^2-y^2)^{\frac{1}{2}}} \pm \sqrt{1 + \frac{(c^2-x^2)^2 y^2}{4x^2(y_1^2-y^2)}}$ .... (c.)

If $x=0$, $\frac{dy}{dx} = 0$, or ∞ $\therefore$ one curve is horizontal and the other is a straight vertical line.

If $y=0$, $\frac{dy}{dx} = \pm 1$ $\therefore$ one curve makes an angle of $+45°$ and the other of $-45°$ with the neutral axis OX.

MOMENT OF RESISTANCE.

$$M = \int p \cdot z\,dy \cdot y = \int p\,z\,y\,dy = \int \frac{p}{y} y^2 z\,dy$$

but $\frac{p}{y}$ is constant $\therefore M = \frac{p}{y}\int y^2 z\,dy = \frac{pI}{y}$ .... (d.)

also $\frac{p}{y} = \frac{f}{y_1}$, $\therefore M_o = \frac{f}{y_1} I$ .... (2 A)

To get equation (7) see Notes, p 67, X.

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p.254
§163 The table on this page is readily obtained from the one on p. 66, Notes.

Ex. VIII.

EXACT SOLUTION.

In this section take moments around AB. Here $CG = \frac{h_2}{2}$ and $DG = h_2 + \frac{h_1}{2}$.

Hence $A_2\frac{h_2}{2} + A_1\left(h_2 + \frac{h_1}{2}\right) = Ay_1$

or $y_1 = \frac{A_2 h_2 + A_1(2h_2 + h_1)}{2A}$

$= \frac{(A_1+A_2)h_2 + A_1(h_2+h_1)}{2A}$

$\therefore y_1 = \frac{Ah_2 + A_1(h_2+h_1)}{2A} = \frac{(h-h_1)A + (h_2+h_1)A_1}{2A} = \frac{h}{2} + \frac{A_1(h_2+h_1) - h_1 A}{2A}$

$y_1 = \frac{h}{2} + \frac{h_2 A_1 - h_1 A_2}{2A}$ .... (1)st.

Again, $I = \Sigma I' + \Sigma y^2 A'$. In this case $\Sigma I' = \frac{1}{12}(A_1 h_1^2 + A_2 h_2^2)$

Also, wt. A_1 : wt. A_2 :: EC : ED whence $A_1 + A_2 : A_1$ :: EC + ED : EC.

or $A_1 + A_2 : A_1 :: \frac{h_1+h_2}{2} : EC = \frac{A_1(h_1+h_2)}{2A}$,

and $A_1 + A_2 : A_2 ::$ EC + ED : $ED = \frac{A_2(h_1+h_2)}{2A}$;

p. 255 §163 Cont.

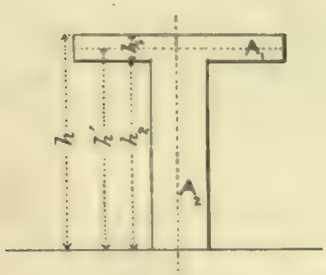
hence $\Sigma y^2 A' = A_2\left[\frac{A_1(h_1+h_2)}{2A}\right]^2 + A_1\left[\frac{A_2(h_1+h_2)}{2A}\right]^2$

$$= \frac{A_1A_2(h_1+h_2)^2(A_1+A_2)}{4A^2} = \frac{A_1A_2(h_1+h_2)^2}{4A}$$

$$\therefore\ I = \frac{A_1h_1^2 + A_2h_2^2}{12} + \frac{A_1A_2(h_1+h_2)^2}{4A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)^{2nd}$$

Ex. VIII.

APPROXIMATE SOLUTION.



$$h' = h_2 + \frac{h_1}{2}$$
$$h = h' + \frac{h_1}{2}$$
$$h_2 = h' - \frac{h_1}{2}$$

$$\left\{ y_1 = \frac{h}{2} + \frac{h_2A_1 - h_1A_2}{2A} \right\}$$

$$\therefore\ y_1 = \frac{h'}{2} + \frac{h_1}{4} + \frac{h'A_1 - \frac{1}{2}h_1A_1 - h_1A_2}{2A}$$

$$= \frac{h'}{2} + \frac{h_1}{4} + \frac{h'}{2}\cdot\frac{A_1}{A} - \frac{h_1}{4}\frac{A_1}{A} - \frac{h_1}{2}\frac{A_2}{A}$$

$$= \frac{h'}{2} + \frac{h_1}{4} + \frac{h'}{2}\frac{A_1}{A} - \frac{h_1}{4}\frac{A_1+A_2}{A} - \frac{h_1}{4}\frac{A_2}{A}$$

Cancelling the common term and neglecting the last term which is very small we have, $y_1 = \frac{h'}{2}\left(1 - \frac{A_1}{A}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)^{1st}$

To find I. $\quad \frac{A_1h_1^2 + A_2h_2^2}{12} = \frac{1}{12}\left(A_1h_1^2 + A_2\left(h' - \frac{h_1}{2}\right)^2\right)$

$$= \frac{1}{12}\left(A_1h_1^2 + A_2h'^2 - A_2h'h_1 + \frac{1}{4}A_2h_1^2\right) \ldots\ldots\ldots\ldots\ldots (a.)$$

$$\frac{A_1}{4A}\cdot A_2\left(h' + \frac{h_1}{2}\right)^2 = \frac{A_1}{4A}\left(A_2h'^2 + A_2h'h_1 + \frac{1}{4}h_1^2\right) \ldots\ldots\ldots\ldots\ldots (b.)$$

Suppose as an approximation $A = 3A_1$ then,

$$= \frac{1}{12}\left(A_2h'^2 + A_2h'h_1 + \frac{1}{4}A_2h_1^2\right) \ldots\ldots\ldots\ldots\ldots (c.)$$

Adding equations (a) and (c), the term $-A_2h'h_1$ cancels the term $+A_2h'h_1$; hence in adding eqs. (a) and (b) to get the value I the term $-\frac{1}{12}\cdot A_2h'h_1$ approx. cancels $+\frac{A_1}{4A}\cdot A_2h'h_1$; and neglecting the terms involving h_1^2 which are small we have,

$$I = h'^2\left(\frac{A_2}{12} + \frac{A_1A_2}{4A}\right); \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)^{2d}$$

$$M_0 = \frac{fI}{y_1} = \frac{f\,h'^2\left(\frac{A_2}{12} + \frac{A_1A_2}{4A}\right)}{\frac{h'}{2}\left(1 + \frac{A_1}{A}\right)} = \frac{fh'}{\frac{1}{2}}\cdot\frac{\frac{A_2A + 3A_1A_2}{12A}}{\frac{A + A_1}{A}} = \frac{fh'}{6}\cdot\frac{A_2A + 3A_1A_2}{A + A_1}$$

but $A = A_1 + A_2$ $\quad \therefore\ M_0 = \frac{fh'}{6}\cdot\frac{(A_1 + A_2 + 3A_1)A_2}{A_1 + A_2 + A_1} = \frac{fh'}{6}\frac{(A_2 + 4A_1)A_2}{A_2 + 2A_1} \ldots\ldots (2.)^{3d}$

.255 Ex. IX   EXACT SOLUTION

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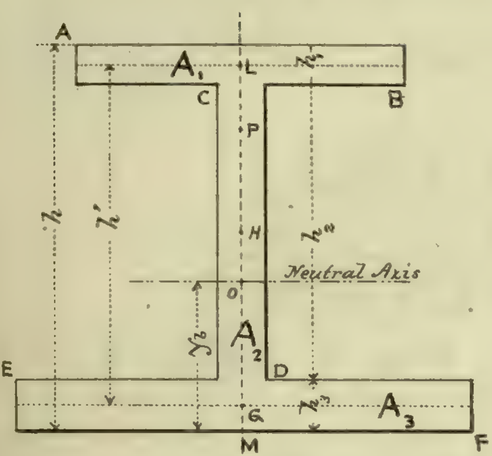


L = centre of gravity of A_1

H = " " " " A_2

G = " " " " A_3

P = " " " " $A_1 + A_2$

O = " " " " $A_1 + A_2 + A_3$

$LH = \dfrac{h_1 + h_2}{2}$ and

$A_1 : A_2 :: PH : PL$ or $A_1 + A_2 : A_1 :: \dfrac{h_1+h_2}{2} : PH$

$$\therefore PH = \frac{A_1(h_1+h_2)}{2(A_1+A_2)}$$

To find C. of G. of $A_1 + A_2$ and A_3.

$$PG = PH + HG = \frac{h_2+h_3}{2} + \frac{A_1(h_1+h_2)}{2(A_1+A_2)}$$

and $A_1 + A_2 : A_3 :: OG : OP$, or $A_1+A_2+A_3 : A_1+A_2 :: PG : OG$,

$$\therefore OG = \frac{(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)}{2A}.$$

Hence

$$MO = y_b = \frac{h_3}{2} + \frac{(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)}{2A} \text{ ; but } \frac{h_3}{2} = \frac{h}{2} - \frac{h_1+h_2}{2};$$

whence

$$y_b = \frac{h}{2} - \frac{(h_1+h_2)A_3 - (h_2+h_3)A_1 - (h_3-h_1)A_2}{2A} \text{ . ...} (3)^{1st}$$

$$\left\{ I = \Sigma I' + \Sigma y^2 A' \right\} \qquad \Sigma I' = \frac{A_1 h_1^2 + A_2 h_2^2 + A_3 h_3^2}{12}$$

For $\Sigma y^2 A'$ we have, $LH = \dfrac{h_1+h_2}{2}$; $HG = \dfrac{h_2+h_3}{2}$ and

$$HG - OG = HO = \frac{h_2+h_3}{2} - \frac{(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)}{2A};$$

$$LO = LH + HO = \frac{h_1+2h_2+h_3}{2} - \frac{(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)}{2A};$$

$$OG = \frac{(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)}{2A}$$

HO, LO, OG are the y's needed. Substitute in $\Sigma y^2 A'$ and

$$\Sigma y^2 A' = A_2\left[\frac{(h_2+h_3)^2}{4} - 2\cdot\frac{h_2+h_3}{2}\left(\frac{(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)}{2A}\right) + \overline{OG}^2\right]$$

$$+ A_1\left[\frac{(h_1+2h_2+h_3)^2}{4} - 2\cdot\frac{h_1+2h_2+h_3}{2}\left(\frac{(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)}{2A}\right) + \overline{OG}^2\right]$$

$$+ A_3 \cdot \overline{OG}^2$$

$$\therefore \Sigma y^2 A' \left\{\begin{aligned} = & \frac{A_2}{4A}\left[(h_2+h_3)^2(A_1+A_2+A_3) - 2(h_2+h_3)^2(A_1+A_2) - 2A_1(h_1+h_2)(h_2+h_3)\right] + A_2\cdot\overline{OG}^2 \\ + & \frac{A_1}{4A}\left[(h_1+2h_2+h_3)^2(A_1+A_2+A_3) - 2(h_1+2h_2+h_3)(h_2+h_3)(A_1+A_2) - 2A_1(h_1+2h_2+h_3)(h_1+h_2)\right] \\ + & A_1\cdot\overline{OG}^2 + A_3\cdot\overline{OG}^2 \end{aligned}\right\} (a)$$

Now the terms involving $\overline{OG}^2$ when collected are,

$$A\cdot\overline{OG}^2 = \frac{\left[(A_1+A_2)(h_2+h_3) + A_1(h_1+h_2)\right]^2}{4A} \text{ ....} (b.)$$

p.255 Also the first term in eq.(a) is

§163 $$= \frac{A_2}{4A}\left[(h_1+h_3)^2A_3 - (h_2+h_3)^2(A_1+A_2) - 2A_1(h_1+h_2)(h_2+h_3)\right]$$

Cont. Hence eq.(a) may be written, if we perform the operations indicated in eq.(b), as follows-

$$\begin{aligned}\Sigma y^2A' = {} & \frac{A_2}{4A}\left[(h_2+h_3)^2A_3 - (h_2+h_3)^2(A_1+A_2) - 2A_1(h_1+h_2)(h_2+h_3)\right] \\ & + \frac{A_1}{4A}\Big[(h_1+2h_2+h_3)^2(A_1+A_2+A_3) - 2A_1(h_1+2h_2+h_3)(h_1+h_2) \\ & \qquad - 2(h_1+2h_2+h_3)(h_2+h_3)(A_1+A_2)\Big] \\ & + \frac{1}{4A}\left[(A_1+A_2)^2(h_2+h_3)^2 + 2A_1(A_1+A_2)(h_2+h_3)(h_1+h_2) + A_1^2(h_1+h_2)^2\right]\end{aligned}$$

Multiplying both sides by $4A$ and remembering that

$$(h_1+2h_2+h_3)^2 = [(h_1+h_2)+(h_2+h_3)]^2 \quad \text{we have,}$$

$$\begin{aligned}4A\cdot\Sigma y^2A' = {} & A_2A_3(h_2+h_3)^2 - A_2(h_2+h_3)^2 - A_1A_2(h_2+h_3)^2 - 2A_1A_2(h_1+h_2)(h_2+h_3) \\ & + A_1(A_1+A_2)[(h_1+h_2)+(h_2+h_3)]^2 + A_1A_3(h_1+2h_2+h_3)^2 - 2A_1(A_1+A_2)[(h_2+h_3)^2 \\ & + (h_1+h_2)(h_2+h_3)] - 2A_1^2[(h_1+h_2)^2+(h_2+h_3)(h_1+h_2)] + A_1^2(h_2+h_3)^2 + 2A_1A_2(h_2+h_3)^2 \\ & + A_2^2(h_2+h_3)^2 + 2A_1(A_1+A_2)(h_2+h_3)(h_1+h_2) + A_1^2(h_1+h_2)^2\end{aligned}$$

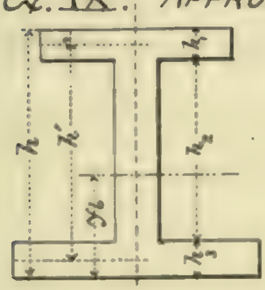
Collecting and cancelling some of the terms and expanding others.

$$\begin{aligned}4A\cdot\Sigma y^2A' = {} & A_2A_3(h_2+h_3)^2 + A_1A_3(h_1+2h_2+h_3)^2 + A_1A_2(h_2+h_3)^2 - 2A_1A_2(h_1+h_2)(h_2+h_3) \\ & + A_1^2(h_1+h_2)^2 + 2A_1^2(h_1+h_2)(h_2+h_3) + A_1^2(h_2+h_3)^2 + A_1A_2(h_1+h_2)^2 \\ & + 2A_1A_2(h_1+h_2)(h_2+h_3) + A_1A_2(h_2+h_3)^2 - 2A_1^2(h_2+h_3)^2 - 2A_1A_2(h_2+h_3)^2 \\ & - 2A_1^2(h_1+h_2)^2 - 2A_1^2(h_2+h_3)(h_1+h_2) + A_1^2(h_2+h_3)^2 + A_1^2(h_1+h_2)^2 \\ = {} & A_2A_3(h_2+h_3)^2 + A_1A_3(h_1+2h_2+h_3)^2 + A_1A_2(h_1+h_2)^2\end{aligned}$$

Hence $$\Sigma y^2A' = \frac{1}{4A}\left\{A_1A_3(h_1+h_3+2h_2)^2 + A_1A_2(h_1+h_2)^2 + A_2A_3(h_2+h_3)^2\right\}$$

$$\therefore I = \frac{A_1h_1^2 + A_2h_2^2 + A_3h_3^2}{12} + \frac{1}{4A}\left\{A_1A_3(h_1+h_3+2h_2)^2 + A_1A_2(h_1+h_2)^2 + A_2A_3(h_2+h_3)^2\right\} \quad (3)^{2nd}$$

Ex. IX. APPROXIMATE SOLUTION. $h' = h_2 + \frac{h_1+h_3}{2}$



$$\left\{ y_b = \frac{h}{2} - \frac{(h_2+h_1)A_3 - (h_3+h_2)A_1 - (h_3-h_1)A_2}{2A} \right\}$$

$$= \frac{h'}{2} - \frac{h'A_3}{2A} + \frac{h'A_1}{2A} + \frac{0\cdot A_2}{2A} = \frac{h'}{2}\left(1 - \frac{A_3 - A_1}{A}\right)$$

$$\therefore y_b = \frac{h'}{2}\cdot\frac{A_1+A_2+A_3-A_3+A_1}{A} = \frac{h'}{2}\cdot\frac{A_2+2A_1}{A} \quad\ldots\ldots\ldots\ldots (4)^{1st}$$

p.256 To find the value of I from the equation on p.76 we neglect
§163 terms involving h_1^2, h_3^2 and remembering that $h_1 + h_2 = h'$, nearly;
Cont. $h_2 + h_3 = h'$, nearly; $h_1 + h_3 + 2h_2 = 2h'$, nearly; we have the equation in the text.

The value of M_o presents no difficulty.

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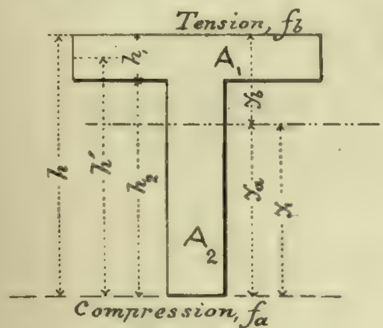
p.257 CASE I. $f_a > f_b$

§164



From eq.(2) p.255 $\left\{ y_a = \frac{h'}{2}\left[\frac{A+2A_1}{A}\right]\right\}$

but $A = A_1 + A_2$ and $h' = y_a + y_b$

$$\therefore y_a = \frac{y_a + y_b}{2}\left(\frac{A_2 + 2A_1}{A_1 + A_2}\right)$$

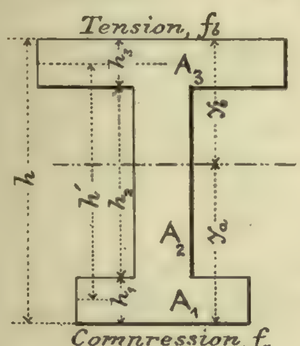
$$\therefore A_1 y_a + A_2 y_a = \frac{A_2 y_a}{2} + \frac{A_2 y_b}{2} + A_1 y_a + A_1 y_b \quad \therefore$$

$$A_1 = \frac{A_2}{2}\cdot\frac{y_a - y_b}{y_b} = A_2\left(\frac{f_a - f_b}{2f_b}\right) \text{.........................} (2.)$$

Also

$$M_o = \frac{f_a h'}{6}\left[\frac{\left(A_2 + 2A_2\cdot\frac{f_a - f_b}{f_b}\right)A_2}{A_2 + A_2\cdot\frac{f_a - f_b}{f_b}}\right] = \frac{A_2 h'}{6}\cdot(2f_a - f_b) \text{.........................} (3.)$$

CASE II. $f_a > f_b$



$A = A_1 + A_2 + A_3$ and $h' = y_a + y_b$

The first of eq's.(4) p.256 is $\left\{ y_b = \frac{h'}{2}\cdot\frac{A_2 + 2A_1}{A}\right\}$

$\therefore y_b = \frac{y_a + y_b}{2}\cdot\frac{A_2 + 2A_1}{A_1 + A_2 + A_3}$; reducing we get

$$A_3 = \frac{y_a}{y_b}A_1 + \frac{A_2}{2y_b}(y_a - y_b) = \frac{f_a}{f_b}A_1 + \frac{f_a - f_b}{2f_b}\cdot A_2 \; ; \text{..........} (4.)$$

The last of eq's.(4) p.256 is

$$\left\{ M_o = \frac{f_b h'}{6}\cdot\frac{A_2(A_2 + 4A_1 + 4A_3) + 12A_1A_3}{A_2 + 2A_1}\right\}$$

Substituting the value of A_3 from (4), and reducing, we get

$$M_o = h'\left[\frac{10A_1A_2f_a - A_2^2 f_b - 2A_1A_2f_b + 2A_2^2 f_a + 12A_1^2 f_a}{6(A_2 + 2A_1)}\right]$$

$$= h'\,\frac{6f_aA_1A_2 + 12A_1^2 f_a + A_2^2(2f_a - f_b) + 2A_1A_2(2f_a - f_b)}{6(A_2 + 2A_1)}$$

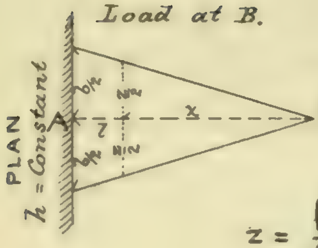
$$= h'\,\frac{f_aA_1(6A_2 + 12A_1) + A_2(A_2 + 2A_1)(2f_a - f_b)}{6(A_2 + 2A_1)}$$

$$\therefore M_o = h'\left\{f_aA_1 + (2f_a - f_b)\frac{A_2}{6}\right\} \text{ similarly also } = h'\left\{f_bA_3 - (f_a - 2f_b)\frac{A_2}{6}\right\} \text{....} (5.)$$

p. 260 § 165

I

Load at B.

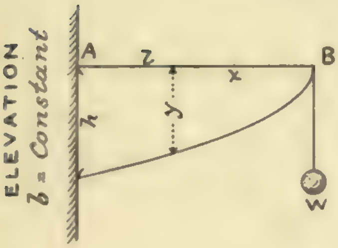


$$M = Wx = \tfrac{1}{6} f z h^2$$

$$M_o = Wl = \tfrac{1}{6} f b h^2$$

$$\therefore \frac{fbh^2}{6l} = \frac{fzh^2}{6x} \quad \therefore$$

$$z = \frac{b}{l}x \quad \therefore \frac{z}{2} = \frac{\frac{b}{2}}{l}x.$$



$$M = Wx = \tfrac{1}{6} f b y^2$$

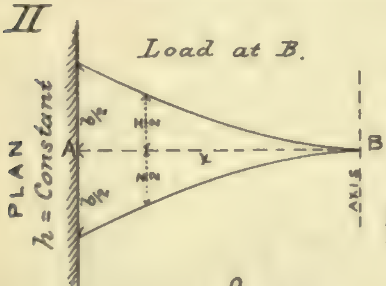
$$M_o = Wl = \tfrac{1}{6} f b h^2$$

$$\therefore \frac{fbh^2}{6l}x = \frac{fby^2}{6}$$

$$\therefore y^2 = \frac{h^2}{l}x.$$

II

Load at B.



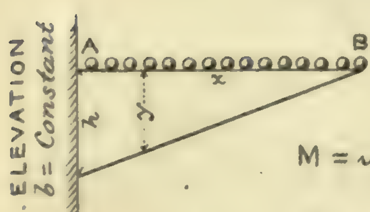
$$M = wx \cdot \frac{x}{2} = \tfrac{1}{6} f z h^2$$

$$M_o = wl \cdot \frac{l}{2} = \tfrac{1}{6} f b h^2$$

$$\frac{fbh^2}{3l^2} \cdot x^2 = \frac{fzh^2}{3}$$

$$\therefore \frac{b}{l^2}x^2 = z \quad \therefore \frac{\frac{b}{2}}{l^2}x^2 = \frac{z}{2}$$

$$\therefore x^2 = \frac{2l^2}{b} \cdot \frac{z}{2}$$



$$M = wx \cdot \frac{x}{2} = \tfrac{1}{6} f b y^2$$

$$M_o = wl \cdot \frac{l}{2} = \tfrac{1}{6} f b h^2$$

$$\frac{fbh^2}{3l^2} \cdot x^2 = \frac{fby^2}{3} \quad \therefore$$

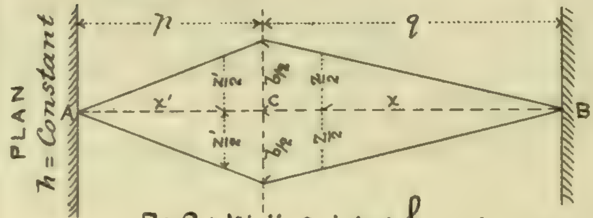
$$\frac{h^2}{l^2}x^2 = y^2$$

$$\therefore y = \frac{h}{l}x$$

III

Component of W at $\begin{cases} A = P \\ B = Q. \end{cases}$

Load at C.



$$P : Q : W :: q : p : l \quad \therefore$$

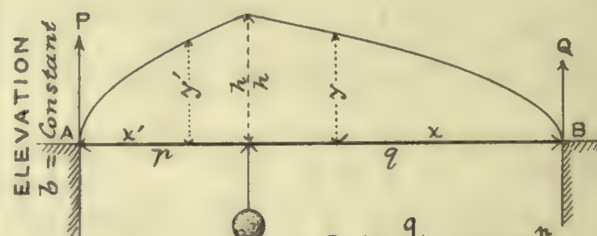
$$P = W\frac{q}{l}, \text{ and } Q = W\frac{p}{l}.$$

$$M = Px' = \tfrac{1}{6} f z' h^2 \;;\; M_o = Pp = \tfrac{1}{6} f b h^2$$

$$\therefore \frac{fbh^2}{6p}x' = \frac{fz'h^2}{6}$$

$$\therefore z' = \frac{b}{p}x' \quad \therefore \frac{z'}{2} = \frac{b/2}{p}x'$$

Similarly, $\quad z = \dfrac{b}{q}x \quad \therefore \dfrac{z}{2} = \dfrac{b/2}{q}x$



$$P = W\frac{q}{l} \;;\; Q = W\frac{p}{l}.$$

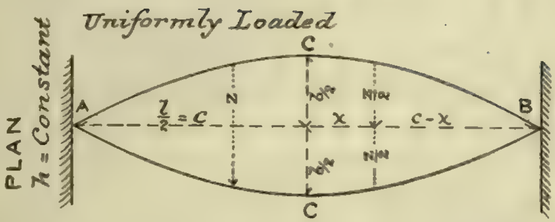
$$M = Px' = \tfrac{1}{6} f b y'^2 \;;\; M_o = Pp = \tfrac{1}{6} f b h^2$$

$$\therefore \frac{fbh^2}{6p} \cdot x' = \frac{fby'^2}{6}$$

$$\therefore y'^2 = \frac{h^2}{p}x' \;;\; \text{similarly } y^2 = \frac{h^2}{q}x.$$

p. 260 § 165 Cont.

IV



$$M = wc(c-x) - w(c-x)\frac{c-x}{2};$$

$$\therefore M = \frac{w}{2}(c^2-x^2) = \frac{1}{6} f z h^2;$$

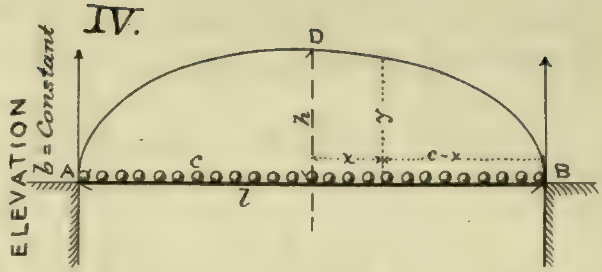
$$M_0 = \frac{wc^2}{2} = \frac{1}{6} f b h^2,$$

$$\therefore \frac{fbh^2}{3c^2} = \frac{fzh^2}{3(c^2-x^2)},$$

$$\therefore \frac{b}{c^2} = \frac{z}{c^2-x^2} \therefore \frac{b}{2}c^2 = \frac{\frac{z}{2}}{c^2-x^2} \therefore \frac{b(c^2-x^2)}{2c^2} = \frac{z}{2};$$

$$\therefore \frac{bc^2}{2c^2} - \frac{bx^2}{2c^2} = \frac{z}{2} \therefore x^2 = \frac{2c^2}{b}\left(\frac{b}{2} - \frac{z}{2}\right).$$

IV.



$$M = wc(c-x) - w(c-x)\frac{c-x}{2}$$

$$\therefore M = \frac{w}{2}(c^2-x^2) = \frac{1}{6} f b y^2$$

$$M_0 = \frac{wc^2}{2} = \frac{1}{6} f b h^2,$$

$$\therefore \frac{fbh^2}{3c^2} = \frac{fby^2}{3(c^2-x^2)},$$

$$\therefore \frac{h^2}{c^2} = \frac{y^2}{c^2-x^2} \therefore h^2(c^2-x^2) = c^2y^2,$$

$$\therefore \frac{x^2}{h^2} + \frac{y^2}{c^2} = 1.$$

---

p. 261 § 167 — Let $s_1 W'$ = gross safe load of first beam (whose breadth is b')
$s_2 B'$ = wt. (multiplied by factor of safety) of first beam.
Then $s_1 W' - s_2 B'$ = net. load and

$$\frac{s_1 W'}{s_1 W' - s_2 B'} = \text{ratio of gross to net load}$$

Now the weight of the beam (which is proportional to the vol. $b'h'l'$), the gross load it can carry ($W = \frac{nfb'h'^2}{ml'}$) and the net load, each increase directly as the breadth (b'), when the depth (h') and the length (l') are unchanged.

We desire to increase the net load from $s_1 W' - s_2 B'$ to $s_1 W'$ or in the ratio $\frac{s_1 W'}{s_1 W' - s_2 B'}$. Increase the breadth in this ratio and the wt., gross load, and net load will each be increased in the same ratio. Let the breadth of new beam = b; then

$$b = \frac{b' s_1 W'}{s_1 W' - s_2 B'} \quad \cdots\cdots (1)$$

so if W = new gross load, $W = s_1 W' \frac{s_1 W'}{s_1 W' - s_2 B'} = \frac{s_1^2 W'^2}{s_1 W' - s_2 B'} \cdots (2)$

and the weight of beam multiplied by its factor of safety is

$$s_2 B = s_2 B' \cdot \frac{s_1 W'}{s_1 W' - s_2 B'} \quad \therefore \quad B = \frac{B' s_1 W'}{s_1 W' - s_2 B'} \quad \cdots\cdots (3)$$

Then the net load of new beam is

$$W - s_2 B = \frac{s_1^2 W'^2}{s_1 W' - s_2 B'} - \frac{s_2 B' \cdot s_1 W'}{s_1 W' - s_2 B'} = \frac{s_1 W' - s_2 B'}{s_1 W' - s_2 B'} s_1 W' = s_1 W'$$

p. 263 From p. 253 eq. (6) and p. 246 VI we have
§ 167

$$\left\{ M_0 = mWl = nfbh^2 = nfhA \quad \text{and } m = \tfrac{1}{8} \right\}$$

$$\therefore \tfrac{1}{8} Wl = nfhA \quad \therefore W = \frac{8nfhA}{l} \quad \therefore W = s_1 W' + s_2 B = \frac{8nfhA}{l} \quad \text{.......... (7.)}$$

$$B = kw'lA \quad \text{.. (8.)}$$

$$\frac{s_2 B}{W} = \frac{s_2 kw'lA}{\frac{8nfhA}{l}} = \frac{s_2 kw'l}{8nfh} \cdot l = \frac{s_2 kw'l^2}{8nfh} \quad \text{.............................. (9.)}$$

$$\{s_2 B = W\} \therefore \frac{s_2 B}{W} = 1 = \frac{s_2 kw'l}{8nfh} \cdot L \quad \therefore L = \frac{8nfh}{s_2 kw'l} = \frac{Wl}{s_2 B} \quad \text{.............. (10.)}$$

$$L : l : L - l :: W : s_2 B : s_1 W'$$

$$\therefore \quad L : \frac{l}{s_2} : \frac{L-l}{s_1} :: W : B : W' \quad \text{.............................. (11.)}$$

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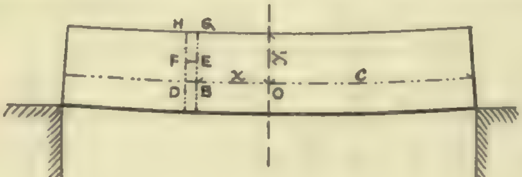
p. 266 DISTRIBUTION OF SHEARING STRESS IN BEAMS (see A.M. § 309)
§ 168



It has been shown, Notes p. 33 top, that shearing stress cannot exist along one plane only, but must be accompanied by a stress of equal intensity on a plane at right angles. Consquently to find the intensity of the shearing stress at any point, as E, on the vertical plane whose trace is BEG, it is sufficient to find the intensity of the shearing stress on the horizontal plane whose trace is EF.

"Let HFD be another vertical section very near GEB. If the moment of flecture at HFD differs from that at GEB, then there must be a corresponding difference in the amount of the direct stress on two corresponding parts of the planes of section, such as GE and HF (In the case shown in the figure, that direct stress is a thrust and is greatest at GE). The difference constitutes a horizontal force acting on the solid HFGE; and in order to maintain the the equilibrium of that solid the amount of shearing stress on the plane FE must be equal and opposite to that horizontal force. That amount being divided by the area of the plane FE gives the intensity of the shearing stress."

Let $\overline{OB} = x$, $BE = y$, $BG = y_1$, $BD = EF$ (sensibly) $= HG$ (sensibly) $= dx$
p = intensity of horizontol stress at E.

p.266 §168 q = intensity of horizontal shearing stress at E;
q_0 = " " " " " " the neutral axis where it is usually a maximum;
z = thickness of beam at E;
z_0 = " " " " neutral axis:

Following the above solution we have
Shearing force on plane $\overline{FE}\cdot z$ =

$$q\,z\,dx = \int_y^{y_1} \frac{dp}{dx}.dx\,.\,z\,dy \quad \text{.......................} \quad (a)$$

From p.252 §162; Notes p.73 eq.(d) $\left\{M = p\frac{I}{y}\right\}$ $\therefore$ $p = \frac{M}{I}y$
and $\frac{dp}{dx}dx = \frac{dM}{I\,dx}dx\cdot y$ $\therefore$

$$q\,z\,dx = \int_y^{y_1} \frac{dM}{I\,dx}dx\,.\,y\,z\,dy = \frac{dM}{I\,dx}dx\int_y^{y_1} y\,z\,dy \quad \text{...............} \quad (b)$$

From p.243, §160, eq.(10) $\left\{M = \int F\,dx\right\}$ $\therefore$ $\frac{dM}{dx} = F$

$$\therefore\ q\,z\,dx = \frac{F\,dx}{I}\int_y^{y_1} y\,z\,dy \qquad \therefore\ q = \frac{F}{Iz}\int_y^{y_1} y\,z\,dy \quad \text{.......................} \quad (c)$$

$$q_0 = \frac{F}{Iz_0}\int_0^{y_1} y\,z\,dy \quad \text{.................................} \quad (2)$$

To find the values of $q_0\frac{A}{F} = \frac{A}{Iz_0}\int_0^{y_1} y\,z\,dy$.

For values of I see table, Notes p.66.

I Rectangle, $z_0 = z = b$; $\int_0^{y_1} z\,y\,dy = b\,\frac{1}{2}y_1^2 = b.\frac{1}{8}h^2$ $\therefore$

$$q_0\frac{A}{F} = \frac{h\,b.\,b\,\frac{1}{8}h^2}{\frac{h^3 b}{12}.\,b} = \frac{3}{2}.$$

II Ellipse or Circle; The equation is $\left\{\frac{(\frac{1}{2}z)^2}{\frac{b^2}{4}} + \frac{y^2}{\frac{h^2}{4}} = 1\right\}$ $\therefore z = 2\sqrt{\frac{b^2}{4} - \frac{b^2}{h^2}y^2}$,

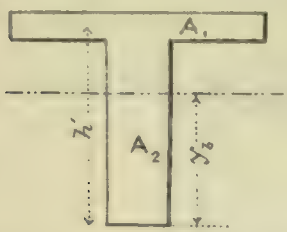
$\int_0^{y_1} z\,y\,dy = 2\int_0^{\frac{h}{2}}\left(\frac{b^2}{4} - \frac{b^2}{h^2}y^2\right)^{\frac{1}{2}} y\,dy = \frac{h^2}{b^2}\int_0^{\frac{h}{2}}\left(\frac{b^2}{4} - \frac{b^2}{h^2}y^2\right)^{\frac{1}{2}}.\,2\frac{b^2}{h^2}y\,dy$, the general integral of which is $-\frac{2}{3}\frac{h^2}{b^2}\left(\frac{b^2}{4} - \frac{b^2}{h^2}y^2\right)^{\frac{3}{2}} + C$; and taking this between the limits 0 and $\frac{h}{2}$; $\int_0^{y_1} z\,y\,dy = \frac{2}{3}\frac{h^2}{b^2}\left(\frac{b}{2}\right)^3 = \frac{1}{12}h^2 b$;

p.267
§ 168
Cont.

$$\therefore\ q_0\frac{A}{F} = \frac{\frac14 \pi h b \cdot \frac1{12} h^2 b}{\frac1{64}\pi h^3 b \cdot b} = \frac43$$

III, IV, V, and VI These are all obtained without difficulty by taking the integrals between the proper limits

VII



In this case $z = z_0$ $\therefore$ $\dfrac{A\int_0^{y_1} z\,y\,dy}{I z_0} = \dfrac{A\int_0^{y_b} y\,dy}{I}$.

$\int_0^{y_1} y\,dy = \frac12 y_1^2$; from eq.(2) p.255 § 163 we have

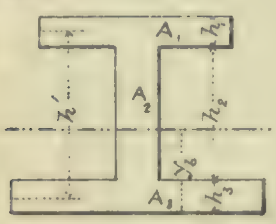
$\left\{y_1 = \frac{h'}{2}\left(1+\frac{A_1}{A}\right)\right\}$ $\therefore \int_0^{y_1} y\,dy = \frac12\left(\frac{h'}{2}\right)^2\left(\frac{A+A_1}{A}\right)^2$

$\left\{I = h'^2\left(\frac{A_2}{12}+\frac{A_1A_2}{4A}\right)\right\}$ $\therefore\ q_0\frac{A}{F} = \dfrac{\dfrac{A\frac18 h'^2 (A+A_1)^2}{A^2}}{h'^2\left(\frac{A_2}{12}+\frac{A_1A_2}{4A}\right)}$

$$= \frac{\frac18 (A+A_1)^2}{\frac1{12}(AA_2+3A_1A_2)} = \frac32\,\frac{(2A_1+A_2)^2}{A_2(A_1+A_2)+3A_1A_2} = \frac32\,\frac{4A_1^2+4A_1A_2+A_2^2}{4A_1A_2+A_2^2} = \frac32\,\frac{4A_1^2+4A_1(A-A_1)+A_2^2}{4A_1A_2+A_2^2}$$

$$= \frac32\cdot\frac{4AA_1+A_2^2}{A_2(4A_1+A_2)}.$$

VIII



$$q_0\frac{A}{F} = \frac{A}{Iz_0}\int_0^{y_1} z\,y\,dy = \frac{A}{I\frac{A_2}{h_2}}\int_0^{(y_b-h_3)}\frac{A_2}{h_2}\cdot y\,dy + \frac{A}{I\frac{A_2}{h_2}}\int_{(y_b-h_3)}^{y_b}\frac{A_3}{h_3}\cdot y\,dy$$

substituting for y_b its value eq.(4) p.256

$$= \frac{A}{I\frac{A_2}{h_2}}\left\{\frac{A_2}{h_2}\cdot\frac12\left(\frac{h'}{2}\cdot\frac{A_2+2A_1}{A}-h_3\right)^2 + \frac{A_3}{h_3}\cdot\frac12\left(\frac{h'}{2}\cdot\frac{A_2+2A_1}{A}\right)^2 - \frac{A_3}{h_3}\cdot\frac12\left(\frac{h'}{2}\cdot\frac{A_2+2A_1}{A}-h_3\right)^2\right\}$$

$$= \frac{A}{2I\frac{A_2}{h_2}}\left\{\left(\frac{A_2}{h_2}-\frac{A_3}{h_3}\right)\left(\frac{h'}{2}\cdot\frac{A_2+2A_1}{A}-h_3\right)^2 + \frac{A_3}{h_3}\left(\frac{h'}{2}\cdot\frac{A_2+2A_1}{A}\right)^2\right\}$$

substituting for I its value eq. (4) p.256

$$= \frac{A\left\{\left(\frac{A_2}{h_2}-\frac{A_3}{h_3}\right)\left(\frac{h'}{2}\cdot\frac{A_2+2A_1}{A}-h_3\right)^2 + \frac{A_3}{h_3}\left(\frac{h'}{2}\cdot\frac{A_2+2A_1}{A}\right)^2\right\}}{\frac{A_2}{h_2}\cdot h'^2\cdot 2\left\{\frac{A_2}{12}+\frac{A_2A_3+A_1A_2+4A_1A_3}{4A}\right\}}$$

reducing the two terms in the denominator to a common denom. remembering that $A = A_1+A_2+A_3$, and transferring the 12 A to the num.

$$= \frac{12A^2\left\{\left(\frac{A_2}{h_2}-\frac{A_3}{h_3}\right)\left(\frac{h'}{2}\cdot\frac{A_2+2A_1}{A}-h_3\right)^2 + \frac{A_3}{h_3}\left(\frac{h'}{2}\cdot\frac{A_2+2A_1}{A}\right)^2\right\}}{\frac{h'^2}{h_2}A_2\left(24A_1A_3+8A_1A_2+8A_2A_3+2A_2^2\right)}$$

p. 268
§ 169

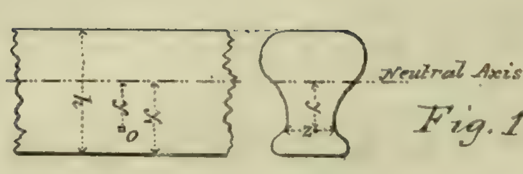


Fig. 1

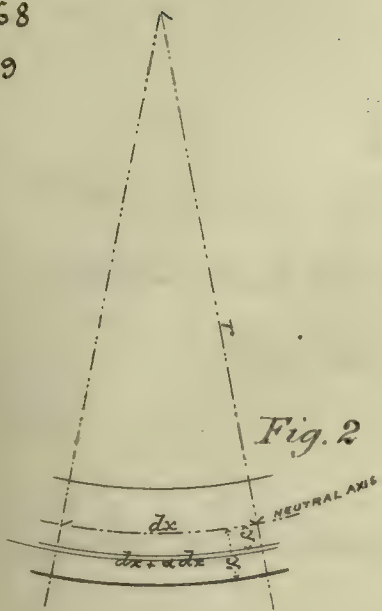


Fig. 2

M = bending moment any where, any load.
M_0 = maximum value of M under proof load.
M_0' = " " " M " any load.
$I = \int y^2 z\, dy$ = moment of inertia at any section.
I_0 = moment of inertia at the point where $M = M_0$.
I_0' = " " " " " " " $M = M_0'$.
p = intensity of stress at any point as O whose distance from the neutral axis is y.
f = stress at surface = max. value of p at any one Xsection.
f' = modulus of proof strength = $\frac{1}{2}$ to $\frac{1}{3}$ of ultimate strength.
f_0' = stress, at surface, at cross-section where $M = M_0'$.
y_1 = distance from neutral axis to the weakest side of the beam.
$y_1 = m'h$ where h is the depth of the beam and m' a numerical factor depending on the shape of the cross-section.
i = inclination of the beam at any point.
i_1' = value of i, at point of greatest slope under any load, W.
i_1 = " " i " " " " " " proof load
v, v_1' and v_1 are the values of the deflection corresponding to i, i_1' and i_1

From pp. 226, 227 § 149 we have $\left\{ \alpha = \frac{\Delta x}{x} = \frac{p}{\mathcal{E}} \right\}$ where α is the intensity of the strain and $\mathcal{E}$ the modulus of direct elasticity, or that force which if applied to a body would stretch it to double its length or crush it to zero, if the elasticity did not change.

From Fig. 2 $\quad dx : (1+\alpha)dx :: r : r + y$

$$\therefore 1 : 1+\alpha :: r : r+y \quad \therefore r + r\alpha = r + y$$

$\therefore r = \frac{y}{\alpha}$ and $\frac{1}{r} = \frac{\alpha}{y} = \frac{p}{\mathcal{E}y}$; but from Notes, p. 73 Eq. (d),
$M = \frac{pI}{y} \quad \therefore \frac{p}{y} = \frac{M}{I} = \frac{f}{y_1}$, a constant at any section.

$$\therefore \frac{1}{r} = \frac{p}{\mathcal{E}y} = \frac{p_1}{\mathcal{E}y_1} = \frac{f}{\mathcal{E}y_1} = \frac{M}{\mathcal{E}I} \quad \text{.................} \quad (1)$$

For the conditions mentioned in the text these equations

p.268 §169 become, $\frac{1}{r_0} = \frac{f'}{\mathcal{E}y_1} = \frac{f'}{\mathcal{E}m'h} = \frac{M_0}{\mathcal{E}I_0}$ .... (2).

From p. 256, § 164, Eq.(1) $\left\{ f_a + f_b : f_a : f_b :: h : y_a : y_b \right\}$

$$\therefore \frac{1}{r_0} = \frac{f_a'}{\mathcal{E}y_a} = \frac{f_a' + f_b'}{\mathcal{E}h} \text{ .... (2 A).}$$

From Eq.(2) we have $\frac{f'}{y_1} = \frac{M_0}{I_0}$ ∴ if we multiply the last member of Eq.(1) by $\frac{f'}{y_1}$ and divide by its equal $\frac{M_0}{I_0}$ we get

$$\frac{1}{r} = \frac{p}{\mathcal{E}y} = \frac{p_1}{\mathcal{E}y_1} = \frac{f}{\mathcal{E}y_1} = \frac{f'}{\mathcal{E}y_1} \cdot \frac{MI_0}{M_0 I} \text{ .... (a).}$$

Step II, Slope. $\tan i = \frac{dv}{dx}$; but since i is always very small $i = \frac{dv}{dx}$.

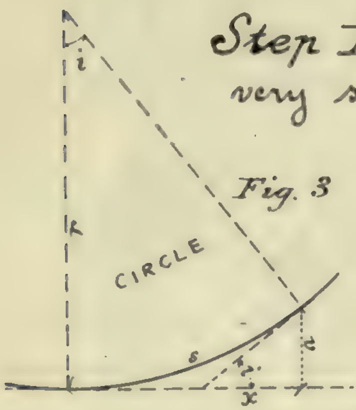


Fig. 3

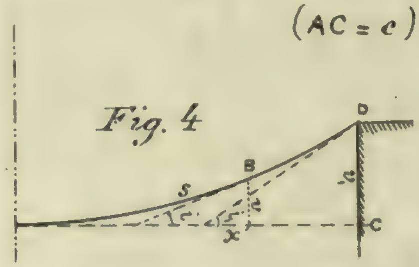


Fig. 4

(AC = c)

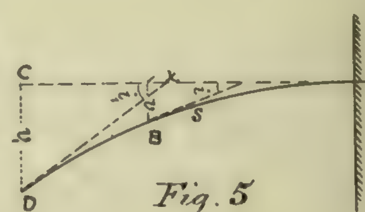


Fig. 5

The value of the radius of curvature, from the Calculus, is

$$r = \frac{\frac{ds^3}{dx^3}}{\frac{d^2v}{dx^2}} \quad \therefore \quad \frac{1}{r} = \frac{\frac{d^2v}{dx^2}}{\left(1 + \frac{dv^2}{dx^2}\right)^{3/2}}.$$

When the curvature is very slight $\frac{dv}{dx}$ is a very small term compared with 1 and may be neglected; which gives

$$\frac{1}{r} = \frac{d^2v}{dx^2}, \text{ but } i\,dx = dv \quad \therefore \quad i = \frac{dv}{dx} \text{ .... (9).}$$

$$\therefore \frac{di}{dx} = \frac{d^2v}{dx^2} \quad \therefore \quad \frac{1}{r} = \frac{di}{dx} = \frac{d^2v}{dx^2} \text{ .... (b).}$$

$$\therefore \quad di = \frac{dx}{r} \text{ .... (c).}$$

Steps II and III; Slope and Deflection.

Under any load

$$\left.\begin{aligned} i &= \int \frac{dx}{r} = \int \frac{M}{\mathcal{E}I}\,dx = \frac{1}{\mathcal{E}}\int \frac{M}{I}\,dx \text{ ....} \\ v &= \int i\,dx = \frac{1}{\mathcal{E}}\iint \frac{M}{I}\,dx^2 \end{aligned}\right\} \text{ .... (d).}$$

<u>Under any load</u> W. From Eq. (2 A), p. 252. $\left\{ M_0 = \frac{fI}{y_1} \right\}$; and to conform to the notation of the present article we write

$$M_0' = \frac{f_0' I_0'}{y_1} \quad \therefore \quad \frac{f_0'}{y_1} = \frac{M_0'}{I_0'} = \frac{mWl}{n'bh^3} \text{ .... (e).}$$

p. 271 §169 and we have $i = \int \frac{dx}{r} = \int \frac{M}{\mathcal{E} I} dx = \int \frac{f_0'}{y_1} \cdot \frac{I_0'}{M_0'} \cdot \frac{M}{\mathcal{E} I} dx = \frac{f_0'}{\mathcal{E} y_1} \int \frac{M I_0'}{M_0' I} dx$ ...... (f).

$$v = \int i\, dx = \frac{f_0'}{\mathcal{E} y_1} \iint \frac{M I_0'}{M_0' I} dx^2 \quad \text{..................(g).}$$

The quantity $\frac{M I_0'}{M_0' I}$ is a numerical ratio depending on the mode of distribution of the loading and supporting forces and the mode of variation of the cross-section of the beam. Hence it is evident that we must have the complete integrals

$$\int_0^c \frac{M I_0'}{M_0' I} dx = m'' c \quad ; \quad \int_0^c \int_0^x \frac{M I_0'}{M_0' I} dx^2 = n'' c^2 .$$

From Eqs. 3 & 6, pp. 252, 253, $\frac{f_0'}{y_1} = \frac{M_0'}{I_0'} = \frac{m W l}{n' b h^3}$

$$\therefore \; i_1' = \frac{f_0'}{\mathcal{E} y_1} \int_0^c \frac{M I_0'}{M_0' I} dx = \frac{m W l}{\mathcal{E} n' b h^3} m'' c = \frac{m''' W c^2}{\mathcal{E} n' b h^3} \; ; \; (m''' = m m'', \text{or} = 2 m m'') \quad (6)$$

also $$v_1' = \frac{f_0'}{\mathcal{E} y_1} \int_0^c \int_0^x \frac{M I_0'}{M_0' I} dx^2 = \frac{m W l}{\mathcal{E} n' b h^3} n'' c^2 = \frac{n''' W c^3}{\mathcal{E} n' b h^3} \; ; \; (n''' = m n'', \text{or} = 2 m n'') \quad (12).$$

<u>Under Proof Load</u>; $$i_1 = \int_0^c \frac{dx}{r} = \int_0^c \frac{M}{\mathcal{E} I} dx = \frac{f'}{\mathcal{E} y_1} \int_0^c \frac{M I_0}{M_0 I} dx ;$$

$$v_1 = \int_0^c i\, dx = \frac{f'}{\mathcal{E} y_1} \int_0^c \int_0^x \frac{M I_0}{M_0 I} dx^2 .$$

The quantity $\frac{M I_0}{M_0 I}$ is again a numerical ratio, and if the load is distributed in the same way as when "<u>under any load W</u>", we will have $\frac{M I_0}{M_0 I} = \frac{M I_0'}{M_0' I}$.

and therefore $$\int_0^c \frac{M I_0}{M_0 I} dx = m'' c \quad ; \quad \int_0^c \int_0^x \frac{M I_0}{M_0 I} dx^2 = n'' c^2$$

Whence the above values become

$$i_1 = \frac{f' m'' c}{\mathcal{E} m' h} \quad \text{.................... (7).}$$

$$v_1 = \frac{f' n'' c^2}{\mathcal{E} m' h} \quad \text{.................... (13).}$$

In sections of equal strength $\frac{f'}{\mathcal{E} m' h} = \frac{f_a'}{\mathcal{E} y_a} = \frac{f_a' + f_b'}{\mathcal{E} h}$

$$\therefore \; i_1 = \frac{m'' (f_a' + f_b') c}{\mathcal{E} h} \quad ; \quad v_1 = \frac{n'' (f_a' + f_b') c^2}{\mathcal{E} h} \quad \text{...... ....(7A) \& (13A).}$$

p.274 §169 In the table on p.274 the values of m'' and n'' are found from the equations

$$\int_0^c \frac{M I_0'}{M_0' I}\,dx = m'' c^2 \quad ; \quad \int_0^c\!\int_0^x \frac{M I_0'}{M_0' I}\,dx^2 = n'' c^2 .$$

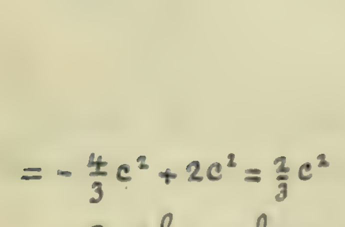
$m''' = m\,m''$; $n''' = m\,n''$ for beams supported at one end.
$m''' = 2\,m\,m''$; $n''' = 2\,m\,n''$ for beams supported at both ends.

A. UNIFORM CROSS-SECTION. Here $\frac{I_0}{I} = 1$ and the values of $\frac{M}{M_0}$ for cases I, II, III, IV, and V can be readily found.

B. UNIFORM STRENGTH AND UNIFORM DEPTH.
Here $M = \frac{fI}{y_1}$ $\therefore$ $\frac{M}{I} = \frac{f}{y_1}$ a constant for uniform strength and uniform depth $\therefore$ $\frac{M I_0'}{M_0' I} = 1$. Since $\frac{1}{r} = \frac{M}{\mathcal{E} I}$, see eq.(1), the curvature is constant.

C. UNIFORM STRENGTH AND UNIFORM BREADTH.
Here f is constant, but $M = \frac{fI}{y_1}$ $\therefore f = \frac{M m' h}{I}$ $\therefore \frac{Mh}{I}$ is constant $\therefore \frac{M I_0'}{M_0' I} = \frac{h_0}{h}$. In using the equations on pp. 78, 79 Notes we must write h for y and h_0 for h.

X Fixed at one end and loaded at the other



$h^2 : h_0^2 :: c - x : c \quad \therefore \frac{h_0}{h} = \sqrt{\frac{c}{c-x}}$

$m'' c = \int_0^c \sqrt{\frac{c}{c-x}}\,.\,dx = 2\sqrt{c}.\sqrt{c} = 2c \quad \therefore m'' = 2$

$n'' c^2 = \int_0^c\!\int_0^x \sqrt{\frac{c}{c-x}}\,.\,dx^2 = \sqrt{c}\int_0^c\!\int_0^x (c-x)^{-\frac{1}{2}}\,dx^2$

but $\int_0^x (c-x)^{-\frac{1}{2}}\,dx = -2(c-x)^{\frac{1}{2}}$ when $x = x$;
when $x = 0$, $\sqrt{c}\int_0^x (c-x)^{-\frac{1}{2}}\,dx = 2\,c^{\frac{1}{2}}.c^{\frac{1}{2}} = 2c$

$$\therefore n'' c^2 = -\int_0^c 2\sqrt{c}\,(c-x)^{\frac{1}{2}}\,dx + \int_0^c 2c\,dx$$

$= -\frac{4}{3}c^2 + 2c^2 = \frac{2}{3}c^2 \quad \therefore n'' = \frac{2}{3}$. $\quad m = 1 \quad \therefore m''' = 2$ and $n''' = \frac{2}{3}$.

Or the above value of n'' may be found, as will be shown on p.8 by remembering that $\int_0^c\!\int_0^x \sqrt{\frac{c}{c-x}}\,.\,dx^2 = \int_0^c \sqrt{\frac{c}{c-x}}\,.(c-x)\,dx$

$\therefore n'' c^2 = \sqrt{c}\int^c (c-x)^{\frac{1}{2}}\,dx = \sqrt{c}\,.\,\frac{2}{3}\,c^{3/2} = \frac{2}{3}c^2 \quad \therefore n'' = \frac{2}{3}$.

Since this last method of integration is easier in the following

p. 274 §169 examples we will show that $\int_0^{x'}\int_0^{x}\varphi(x)\,dx^2 = \int_0^{x'}\varphi(x)\,(x'-x)\,dx$ where $\varphi(x)$ is any function of x and the limits of integration mean that there is to be no constant introduced in the first integration.

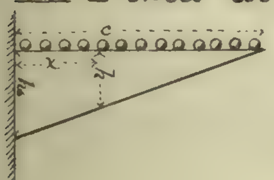
Integrating by parts we have

$$\left\{ \int u\,dv = uv - \int v\,du \right\} \qquad u = \int_0^{x'}\varphi(x)\,dx \;\therefore\; du = \varphi(x)dx$$
$$dv = dx \qquad \therefore v = \int dx = x' \text{ or } x$$

N.B. v has the value x' when the <u>final</u> integration has been performed as in the term uv, while it has the value x when it is still a variable under the integral sign as in the term $\int v\,du$.

$$\therefore \int_0^{x'}\left(\int_0^{x}\varphi(x)\,dx\right)dx = x'\int_0^{x'}\varphi(x)dx - \int_0^{x'}\varphi(x)\,x\,dx,$$
$$= \int_0^{x'}\varphi(x)\,(x'-x)\,dx.$$

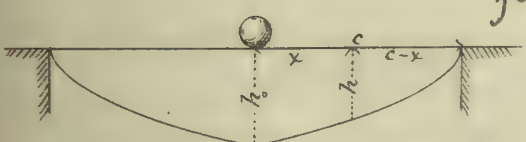
XI Fixed at one end and uniformly loaded.



$$h : h_0 :: c-x : c \quad \therefore \frac{h_0}{h} = \frac{c}{c-x}$$
$$m''c = c\int_0^c \frac{dx}{c-x} = -c\log 0 + c\log c,$$
$$\therefore m'' = \infty \text{ and } m''' = mm'' = \infty.$$
$$n''c^2 = c\int_0^c\int_0^x \frac{dx^2}{c-x} = c\int_0^c \frac{1}{c-x}\cdot(c-x)\,dx = c^2 \quad \therefore n'' = 1, \; n''' = mn'' = \tfrac{1}{2}$$

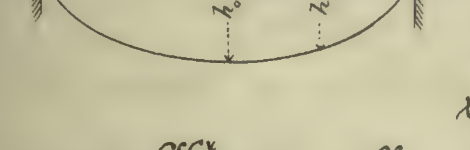
XII Supported at both ends and loaded in the middle

Two equal parabolas and the values are found as in X.



$$m'' = 2, \quad m''' = 2\cdot\tfrac{1}{4}\cdot 2 = 1$$
$$n'' = \tfrac{2}{3}, \quad n''' = 2\cdot\tfrac{1}{4}\cdot\tfrac{2}{3} = \tfrac{1}{3}.$$

XIII Supported at both ends and uniformly loaded



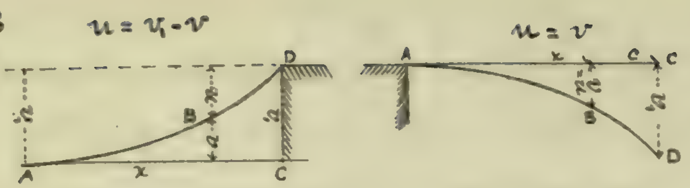
$$h^2 : h_0^2 :: c^2-x^2 : c^2 \quad \therefore \frac{h_0}{h} = \sqrt{\frac{c^2}{c^2-x^2}}$$

$m''c = c\int_0^c \frac{dx}{\sqrt{c^2-x^2}}$ $\therefore m''c = c \text{ arc.sin} \frac{x}{c}$ between the limits 0 and c $= c \text{ arc.sin } 1$ $\therefore m'' = \frac{\pi}{2} = 1.5708$

$n''c^2 = c\int_0^c\int_0^x \frac{dx^2}{\sqrt{c^2-x^2}} = c\int_0^c \frac{c-x}{\sqrt{c^2-x^2}}dx = c\int_0^c \frac{c\,dx}{\sqrt{c^2-x^2}} - c\int_0^c \frac{x\,dx}{\sqrt{c^2-x^2}} = c^2 \text{arc.sin}\frac{x}{c} + c(c^2-x^2)^{\frac{1}{2}} + C$

between the limits 0 and c $= c^2\frac{\pi}{2} - c^2 = (\frac{\pi}{2}-1)c^2 \;\therefore n'' = \frac{\pi}{2} - 1 = 0.5708.$

p. 278
§ 173



SUPPORTED AT BOTH ENDS ; SUP. AT ONE END

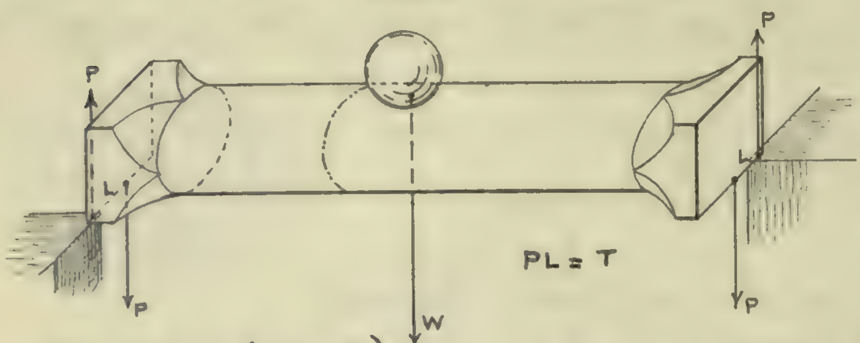
$$\text{Resilience} = \frac{W}{2} v \quad \ldots\ldots\ldots (1)$$

$$\text{Resilience} = \tfrac{1}{2} \int u\, w . dx \quad \ldots (3)$$

$$\left\{ \begin{array}{l} \text{p. 273 § 169 eq. (13)} \sim \; v_1 = \dfrac{n'' f' c^2}{E m' h} \\ \text{p. 252} \\ Wc = n f' b h^2 \quad \therefore W = \dfrac{n f' b h^2}{c} \end{array} \right\}$$

$$\therefore \frac{W}{2} v_1 = \frac{n f' b h^2}{2c} \cdot \frac{n'' f' c^2}{E m' h} = \frac{n n''}{2m'} \frac{f'^2}{E} \cdot c b h \quad \ldots\ldots\ldots (4)$$

p. 280
§ 174



p = horizontal stress from bending moment M.
q = shearing stress at the surface.
r = radius = $\frac{h}{2}$

Notes p. 73 $\left\{ M = \frac{pI}{y} \right\}$ $\therefore p = \frac{My}{I}$. If the beam have a circular cross-section $p = \frac{Mr}{I}$ and from Notes, p. 66, $\left\{ I = \frac{\pi h^4}{64} \right\}$ $\therefore I = \frac{\pi (2r)^4}{64}$ or $I = \frac{\pi r^4}{4}$ $\therefore p = \frac{Mr}{\frac{1}{4}\pi r^4} = \frac{M}{\frac{1}{4}\pi r^3}$ ……………… (a)

The shearing force at the surface due to the twisting moment T is, see Notes p. 35, $\left\{ T = \frac{1}{2} q_1 \pi r_1^3 \right\}$ dropping the suffixes we have $q = \frac{T}{\frac{1}{2}\pi r^3}$; where q = shearing stress at the surface.

II   If $p = q$, then $T = 2M$.

III From eq (3) p. 170 [p. 39 Notes] $\left\{ p_1 = \frac{p_x + p_y}{2} + \sqrt{\frac{(p_x - p_y)^2}{4} + q^2} \right\}$

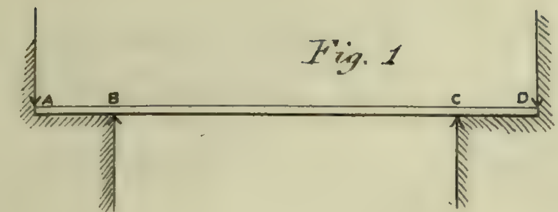
Here $p_x = p$; $p_y = 0$; hence $p_1 = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2}$

$$p_1 = \frac{M}{\frac{1}{2}\pi r^3} + \sqrt{\frac{M^2}{4(\frac{1}{4}\pi r^3)^2} + \frac{T^2}{(\frac{1}{2}\pi r^3)^2}} = \frac{M + \sqrt{M^2 + T^2}}{\frac{1}{2}\pi r^3} = \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4} + \frac{T^2}{4}}}{\frac{1}{4}\pi r^3}$$

$$\therefore p_1 = \frac{M_1}{\frac{1}{4}\pi r^3} = \frac{\frac{M}{2} + \sqrt{\frac{M^2}{4} + \frac{T^2}{4}}}{\frac{1}{4}\pi r^3} \quad \therefore M_1 = \frac{M}{2} + \sqrt{\frac{M^2}{4} + \frac{T^2}{4}} \quad \ldots\ldots (2).$$

Equation (2) is thus shown to be true for a beam of circular cross-section and our author thinks nearly enough true for any other beam. [See table on p. 36 Notes].

p. 283 §176



The fixing of the ends of a beam is equivalent to applying two equal and opposite couples to the two ends as in the figure. The moment thus produced between B and C is constant; p. 249, §.161, Case X.

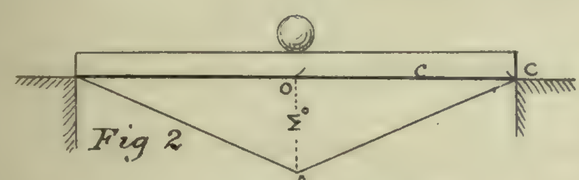
From p. 85, Eq.(f) Notes, $\left\{ i = \int \frac{dx}{r} = \int \frac{M}{\mathcal{E}I} dx \right\}$.

To find the constant moment, M_1, which if the cross-section remained the same throughout as at the point of support, would give an upward inclination equal to i_1';

$$i_1' = \int_0^c \frac{dx}{r} = \int_0^c \frac{M_1}{\mathcal{E}I} dx = \frac{M_1}{\mathcal{E}I} \int_0^c dx = \frac{M_1 c}{\mathcal{E}I} \qquad \therefore M_1 = \frac{\mathcal{E}I i_1'}{c} \quad \text{.......(a).}$$

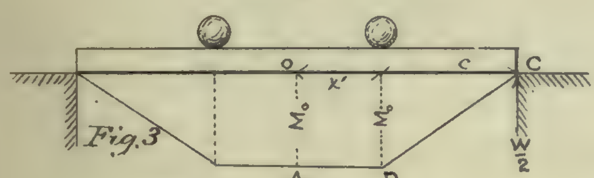
Ex. I Symmetrical load on a beam of uniform X-section, in general

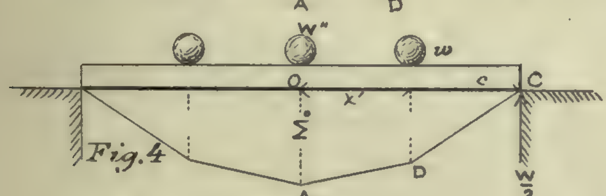
To show that $m'' \geqq \frac{1}{2}$.



$$m''c = \int_0^c \frac{M}{M_0} dx, \text{ since } \frac{I_0}{I} = 1.$$

$$\therefore m''c = \frac{1}{M_0} \int_0^c M dx \quad \text{.........(b)}$$





If we represent the moments graphically as in Figures 2, 3 & 4 then $\int^c M dx$ is represented by the area between the line and the axis of X; viz. in Fig. 2 by AOC, in Fig. 3 by AOCD, and in Fig. 4 by AOCD.

From which it is evident that, when M_0 is the same in each, the area in Fig. 2, AOC, is the smallest of the three.

It is further evident that any more weights applied at other points will give a polygon or curve concave to the axis of X and therefore the area will always be greater than that between a straight line and the axis of X, as in Fig. 2. Q.E.D.

Let v_1' = deflection due to the load W [in eq. 3] acting alone

" v_1'' = " " M_1 acting alone.

From eq.(12) p. 273, § 169, $\left\{ v' = \frac{n''' W c^3}{\mathcal{E} n' b h^3} \right\}$ then $\quad v_1' = \frac{m n'' W l}{\mathcal{E}I} c^2 = \frac{n'' M_0 c^2}{\mathcal{E}I}$,

p.284 where W is the new breaking load and M_0 the moment, at
§176 the middle of the beam, due to this load acting alone

Ext. From eq. (d) p. 84 Notes. $\left\{ v = \frac{1}{E}\iint \frac{M}{I}\,dx^2 \right\}$ $\therefore\ v_1'' = \int_0^c\int_0^x \frac{M_1}{EI}\,dx^2$;

$$\therefore v_1'' = \frac{M_1}{EI}\int_0^c\int_0^x dx^2 = \frac{M_1}{2EI}c^2 .$$

$$\therefore v_1 = v_1' - v_1'' = \frac{n'' M_0 c^2}{EI} - \frac{M_1 c^2}{2EI} . \quad \text{But } M_1 = m'' M_0 \ ;$$

$$\therefore v_1 = \frac{n'' \frac{M_1}{m''} c^2}{EI} - \frac{1}{2}\frac{M_1 c^2}{EI} = \left(\frac{n''}{m''} - \frac{1}{2}\right)\frac{M_1 c^2}{EI} = \left(n'' - \frac{m''}{2}\right)\frac{M_0 c^2}{EI} \quad \text{.............(4.)}$$

The f in equation (5), in text, should be f′; also in equations (7) and (8) the f in the last eq. of each should be f′.

---

p.285 Ex. III
§176 From eq.(4), p.248, $\left\{ M = \frac{w}{2}(c^2 - x^2) \right\}$ [w′ being 0]. But $M_1 = m'' M_0$
$= \frac{1}{6} Wc = \frac{1}{6} wlc = \frac{1}{3} wc^2$

$$\therefore M = M_1 \text{ becomes } \frac{w}{2}(c^2 - x^2) = \frac{1}{3} wc^2 \quad \therefore x = \frac{c}{\sqrt{3}} \quad \text{.............(9.)}$$

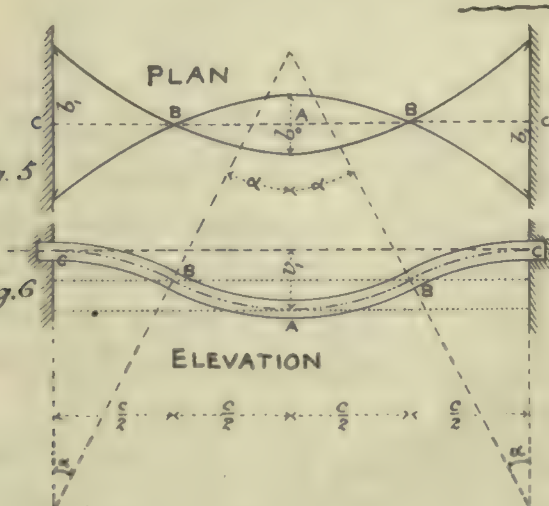


Fig. 5

Fig. 6

Ex. IV *Uniform strength, uniform depth, uniform load.*

Eq. (1) p. 268, § 169, $\left\{ \frac{1}{r} = \frac{M}{EI} \right\}$

p. 73 Notes, § 162, $\left\{ M = \frac{fI}{y_1} \right\}$

In this example y_1 is constant $\therefore \frac{M}{EI} = \frac{1}{r}$ is const.

$$\left\{ \begin{array}{ll} \text{Span } BB = c & w = \frac{W}{2c} \\ \text{Load} = wc = \frac{W}{2c}\cdot c = \frac{W}{2} & \end{array} \right\}$$

$$\left\{ m = \tfrac{1}{8}\ ;\ m'' = 1\ ;\ n'' = \tfrac{1}{2} \right\}$$

Eq. (13), p. 273, § 169, $\left\{ v_1 = \frac{n'' f' c^2}{E m' h} \right\}$ $\therefore\ v_1 = \frac{1}{2}\cdot\frac{n'' f' c^2}{E m' h} = \frac{1}{2}\,\frac{\frac{1}{2} f' c^2}{E m' h} = \frac{1}{4}\cdot\frac{f' c^2}{E m' h}$ ...(10.)

Eq. (16), p. 244, § 160, $\{ M = mWl \}$ $\therefore\ M_0' = \frac{1}{8}\cdot\frac{W}{2}\cdot c = \frac{Wc}{16} = \frac{Wl}{32} = \frac{M_0}{4}$ ....(11.)

$M_0' = M_0 - M_1 \quad \therefore M_1 = M_0 - M_0' = M_0 - \frac{1}{4} M_0 = \frac{3}{4} M_0$

$$\therefore n f b_1 h^2 = M_1 = \tfrac{3}{4} M_0 = \frac{3Wc}{16} = \frac{3Wl}{32} \quad \text{.................(12.)}$$

Eq. (4), p. 248, § 161, $\left\{ M = \frac{w}{2}(c^2 - x^2) \right\}$. When $x = 0$, $M_0 = \frac{wc^2}{2}$ $\therefore$

p.286 §176 Ext.

$$M_1 = \tfrac{3}{4} M_0 = \tfrac{3}{8} w c^2$$

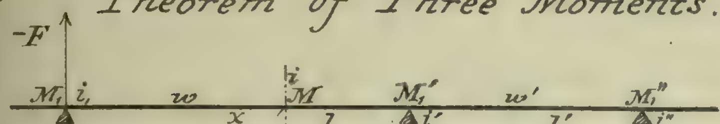
$$M' = M - M_1 = \frac{w(c^2-x^2)}{2} - \frac{3wc^2}{8} = w\left[\frac{4c^2-4x^2-3c^2}{8}\right] = w\cdot\frac{c^2-4x^2}{8}.$$

$$nfb_1h^2 = M_1 = \frac{3wc^2}{8},\ \therefore\ nfh^2 = \frac{3wc^2}{8b_1}\ ;\ nfbh^2 = M' = \frac{w(c^2-4x^2)}{8}\ \therefore nfh^2 = \frac{w(c^2-4x^2)}{8b},$$

$$\left.\begin{array}{l} \therefore \dfrac{3wc^2}{8b_1} = \dfrac{w(c^2-4x^2)}{8b}\ \ \therefore \dfrac{3c^2}{b_1} = \dfrac{c^2-4x^2}{b}\ \ \therefore\ b = \dfrac{c^2-4x^2}{3c^2}b_1 = \dfrac{1}{3}\left(1-\dfrac{4x^2}{c^2}\right)b_1 \\ b_1 : b_0 :: M_1 : M_0' :: \frac{3}{4}M_0 : \frac{1}{4}M_0 :: 3:1\ \ \therefore b_1 = 3b_0\ \ \therefore\ b = \left(1-\dfrac{4x^2}{c^2}\right)b_0 \end{array}\right\} \cdots\cdots(13.)$$

p.287 §178

Theorem of Three Moments. To preface Art. 178



$$m = \int_0^x\!\!\int_0^x w\,dx^2 \ldots\ldots;\quad N = \int_0^l\!\!\int_0^x w\,dx^2\ ;$$

$$m' = \int_0^x m\,dx = \int_0^x\!\!\int_0^x\!\!\int_0^x w\,dx^3\ ;\qquad Q = \int_0^l m\,dx = \int_0^l\!\!\int_0^x\!\!\int_0^x w\,dx^3\ ;$$

$$\int_0^x m'\,dx = \int_0^x\!\!\int_0^x\!\!\int_0^x\!\!\int_0^x w\,dx^4\ ;\qquad K = \int_0^l m'\,dx = \int_0^l\!\!\int_0^x\!\!\int_0^x\!\!\int_0^x w\,dx^4\ .$$

$$\left\{\begin{array}{l} \text{From pp. 83, 84, Notes}\quad \dfrac{1}{r} = \dfrac{M}{EI} = \dfrac{d^2v}{dx^2} \\ \therefore \text{Moment of Resistance} = EI\,\dfrac{d^2v}{dx^2} = M \end{array}\right\}$$

$$EI\,\frac{d^2v}{dx^2} = M_1 + m - Fx \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A.)$$

$$EI\left(\frac{dv}{dx} - \tan i_1\right) = M_1x + \int_0^x m\,dx - \tfrac{1}{2}Fx^2 \ \ldots\ldots\ldots\ldots\ldots\ldots (B.)$$

$$EI\,(v - x\tan i_1) = \tfrac{1}{2}M_1x^2 + \int_0^x m'\,dx - \tfrac{1}{6}Fx^3 \ \ldots\ldots\ldots\ldots\ldots (C.)$$

Make $x = l$, and N, Q, & K the resultant values, respectively, of m, m', & $\int_0^x m'\,dx$. For $x = l$, $v = 0$, $EI\,\frac{d^2v}{dx^2} = M_1'$, and $\frac{dv}{dx} = \tan i_1'$

Equations A, B, & C, will then become, respectively,

$$M_1' = M_1 + N - Fl \ \ldots\ldots\ldots\ldots\ldots\ldots (A'.)$$

$$EI(\tan i_1' - \tan i_1) = M_1l + Q - \tfrac{1}{2}Fl^2 \ \ldots\ldots\ldots\ldots\ldots (B'.)$$

$$-EI\,l\tan i_1 = \tfrac{1}{2}M_1l^2 + K - \tfrac{1}{6}Fl^3 \ \ldots\ldots\ldots\ldots\ldots\ldots (C'.)$$

Eliminating F by subtracting (C′) from (A′); and then (C′) from (B′)

$$\left\{(A')\times\tfrac{1}{6}l^2 \text{ is } \tfrac{l^2}{6}M_1' = \tfrac{l^2}{6}(M_1+N) - \tfrac{1}{6}Fl^3\ ;\ \ (B')\times\tfrac{l}{3} \text{ is } EI\tfrac{l}{3}(\tan i_1' - \tan i_1) = \tfrac{1}{3}M_1l^2 + \tfrac{1}{3}Ql - \tfrac{1}{6}Fl^3 \ldots\ldots\right\}$$

$$M_1'\frac{l^2}{6} + EI\,l\tan i_1 = -\tfrac{1}{3}M_1l^2 + \tfrac{1}{6}Nl^2 - K \ \ldots\ldots\ldots\ldots\ldots (D.)$$

$$EI\,\frac{l}{3}\tan i_1' + \tfrac{2}{3}EI\,l\tan i_1 = -\tfrac{1}{6}M_1l^2 + \tfrac{1}{3}Ql - K \ \ldots\ldots\ldots (E.)$$

p.287 §178 Cont. In equations (D) and (E) N, Q, & K representing the exterior forces are known, but M_1, M_1', $\tan i_1$, $\tan i_1'$, are unknown. These equations, (D) and (E), show that M_1' and $\tan i_1'$ are linear functions of M_1 and $\tan i_1$; whatever the distribution of the external forces may be.

Load uniformly distributed, as is usually the case

$$m = \tfrac{1}{2} w x^2 \;;\quad m' = \int_0^x m\,dx = \tfrac{1}{6} w x^3 \;;\quad \int_0^x m'\,dx = \tfrac{1}{24} w x^4 \;;$$

and making $x = l$,

$$N = \tfrac{1}{2} w l^2 \;;\quad Q = \tfrac{1}{6} w l^3 \;;\quad K = \tfrac{1}{24} w l^4 .$$

Substituting these values in eqs. (D) and (E) and reducing

$$M_1' = -2M_1 - \frac{6\,\mathcal{E}I}{l}\tan i_1 + \tfrac{1}{4} w l^2 \qquad \text{(D')}$$

$$\tan i_1' = -2\tan i_1 - \tfrac{1}{2}\frac{l}{\mathcal{E}I} M_1 + \tfrac{1}{24}\frac{w l^3}{\mathcal{E}I} \qquad \text{(E')}$$

The Moment of Resistance, and Slope, at any support are linear functions of the slope at the abutment, where the beam is supposed not to be fixed and consequently the moment is zero.

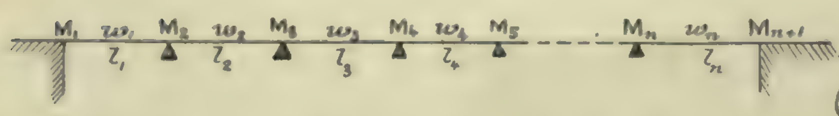
From analogy to eq(D')

$$M_1'' = -2M_1' - \frac{6\,\mathcal{E}I}{l'}\, i_1' + \tfrac{1}{4} w' l'^2 \qquad \text{(G)}$$

$$M_1 = -2M_1' + \frac{6\,\mathcal{E}I}{l}\, i_1' + \tfrac{1}{4} w l^2 \qquad \text{(H.)}$$

It is evident that i_1' will have different signs in the two eqs. (G) and (H). Multiplying (G) by l', (H) by l and adding; we have

$$M_1 l + 2M_1'(l + l') + M_1'' l' - \tfrac{1}{4} w l^3 - \tfrac{1}{4} w' l'^3 = 0 \qquad \text{(I.)}$$



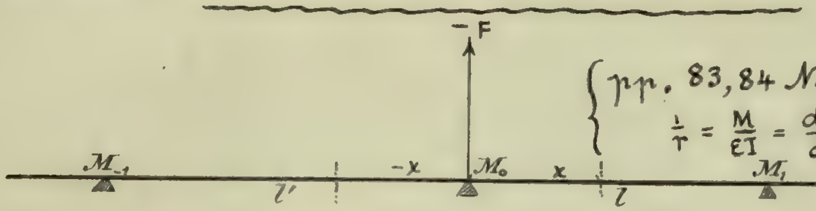
From analogy to eq. (I) we have, remembering that M_1 is $=0$, and M_{n+1} is also $= 0$.

$$\left.\begin{aligned}
2(l_1 + l_2) M_2 + l_2 M_3 - \tfrac{1}{4}(w_1 l_1^3 + w_2 l_2^3) &= 0\\
l_2 M_2 + 2(l_2 + l_3) M_3 + l_3 M_4 - \tfrac{1}{4}(w_2 l_2^3 + w_3 l_3^3) &= 0\\
l_3 M_3 + 2(l_3 + l_4) M_4 + l_4 M_5 - \tfrac{1}{4}(w_3 l_3^3 + w_4 l_4^3) &= 0\\
l_{n-1} M_{n-1} + 2(l_{n-1} + l_n) M_n \quad - \tfrac{1}{4}(w_{n-1} l_{n-1}^3 + w_n l_n^3) &= 0
\end{aligned}\right\} \qquad \text{(J.)}$$

p.287 §178 Cont. All the unknown quantities but one may be made to disappear by multiplying the equations by Indeterminate Coefficients [Besout's method]

p.287 §178

$$\left\{ \begin{array}{l} \text{pp. 83, 84 Notes} \\ \frac{1}{r} = \frac{M}{EI} = \frac{d^2v}{dx^2} \quad \therefore M = EI\frac{d^2v}{dx^2} \end{array} \right\}$$



$$\iint w\,dx^2 = m\ ;\quad \iint \frac{dx^2}{EI} = n\ ;\quad \iint \frac{x\,dx^2}{EI} = q\ ;\quad \iint \frac{dx^2}{EI}m = \iint\frac{dx^2}{EI}\iint w\,dx^2 = V \ldots\ldots (1.)$$

$$M = EI\frac{d^2v}{dx^2} = M_0 - Fx + m \text{ ------------------- } (2.)$$

$$\frac{dv}{dx} - T = M_0\int\frac{dx}{EI} - F\int\frac{x\,dx}{EI} + \int\frac{m\,dx}{EI}$$

$$v - Tx = M_0\iint\frac{dx^2}{EI} - F\iint\frac{x\,dx^2}{EI} + \iint\frac{m}{EI}dx^2 \quad \therefore$$

$$v = Tx - F\iint\frac{x\,dx^2}{EI} + M_0\iint\frac{dx^2}{EI} + \iint m\frac{dx^2}{EI}$$

$$\therefore\ v = Tx - Fq + M_0 n + V \text{ ------------------- } (3)$$

$$\{\text{When } x = l\}\quad M_1 = M_0 - Fl + m_1 \quad \therefore F = \frac{M_0 - M_1 + m_1}{l} \ldots\ldots\ldots (4.)$$

$$v_1 = 0 = Tl - Fq_1 + M_0 n_1 + V_1 \quad \therefore T = \frac{Fq_1 - M_0 n_1 - V_1}{l} \quad \therefore$$

$$T = M_0\left(\frac{q_1}{l^2} - \frac{n_1}{l}\right) - \frac{M_1 q_1}{l^2} + \frac{m_1 q_1}{l^2} - \frac{V_1}{l} \ldots\ldots\ldots (5.)$$

$$-T' = M_0\left(\frac{q_{-1}}{l'^2} - \frac{n_{-1}}{l'}\right) - \frac{M_{-1} q_{-1}}{l'^2} + \frac{m_{-1} q_{-1}}{l'^2} - \frac{V_{-1}}{l'} \ldots\ldots\ldots (5\text{A}.)$$

$$0 = M_0\left(q_1 l'^2 + q_{-1} l^2 - n_1 l l'^2 - n_{-1} l' l^2\right) - M_1 q_1 l'^2 - M_{-1} q_{-1} l^2 +$$
$$m_1 q_1 l'^2 + m_{-1} q_{-1} l^2 - V_1 l l'^2 - V_{-1} l' l^2 - t l^2 l'^2 \text{ ------------------- } (6.)$$

Cross-section uniform and piers equidistant.

To obtain the table on p. 289 we have $\{M' = M - M_1\}$ where $M = \frac{w}{2}(c^2 - x^2)$ for the lightly loaded and $M = \frac{w+w'}{2}(c^2 - x^2)$ for the heavily loaded span.

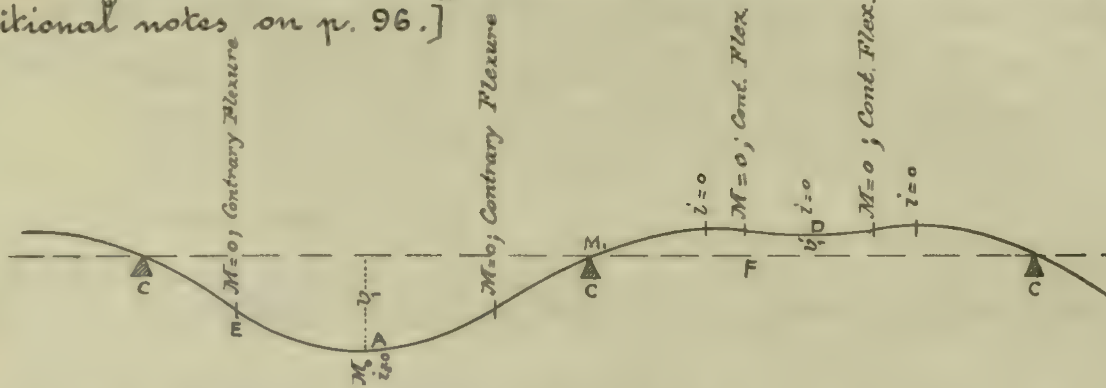
Eq (1) p. 83 Notes $\left\{\frac{1}{r} = \frac{M}{EI}\right\}$; Eq (c) Notes p. 84 $\left\{\frac{1}{r} = \frac{di}{dx}\right\}$

$$\therefore\ di = \frac{dx}{r} = \frac{M}{EI}dx.$$

From which the slope i is found; (for value see text).

☞ [M_1 is greatest when every span is loaded with both w and w' see additional notes on p. 96.]

p. 290
§ 178
Cont.



Since v is positive downwards while i is positive upwards we have $i = -\frac{dv}{dx}$ $\therefore v = -\int i\,dx$ which gives the values, "*Deflection v at any point*", in the table on p. 290; the constant of integration being determined by making $v = 0$ at the pier.

Equations (8) and (9), come by making $\{x' = 0\ ;\ x = 0\}$ in the table.

Central Deflection under Proof Load.

If $\{M_0 > M_1\}$ Eq. (2A) p. 252 $\left\{M_0 = \frac{fI}{y_1}\right\}$ $\therefore M_0 = \frac{f'I}{m'h} = \frac{w + 2w'}{6}c^2$,

$$\therefore I = \frac{(w + 2w')\,m'h\,c^2}{6f'}.$$

Substituting this value in the 2nd. of equations (9)

$$\left\{v_1 = (w + 3w')\frac{c^4}{24EI}\right\} \quad \therefore v_1 = \frac{f'c^2}{4Em'h}\cdot\frac{w + 3w'}{w + 2w'}, \quad \text{.......... (9 A.)}$$

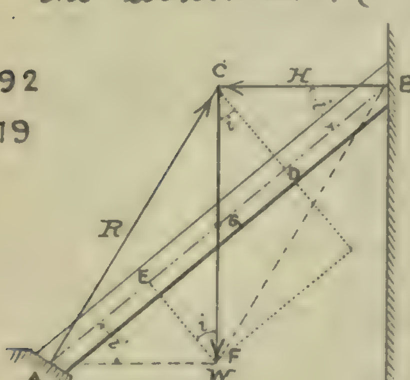
If $\{M_1 > M_0\}$, find the value of I from $M_1 = \frac{f'I}{m'h} = \frac{2w + w'}{6}c^2$

$$\text{which gives} \quad v_1 = \frac{f'c^2}{4Em'h}\cdot\frac{w + 3w'}{2w + w'}, \quad \text{.......... (9 A.)}$$

Points of no curvature, $\frac{1}{r} = \frac{di}{dx} = \frac{M}{EI} = 0 \quad \therefore M = 0$.

The end span must not be longer than CE, nor shorter than CF; and, if the point F does not exist, then the span must not be shorter than a length sufficient to produce the moment M_1 before the end of the bridge will be raised from the abutment. (see text)

p. 292
§ 179



Since the wall is vertical its resistance must be horizontal, W acts through C. of G. of the beam and the resistance of the abutment at A must be such that the three forces shall be in equilibrium

Since $CG = FG$, $\tan CAF = 2\tan i$

p.292 $H = W.\tan ACF = W \div 2\tan i$; $R = \sqrt{H^2+W^2}$ .... (2.)

§179 Cont. Longitudinal component of load $= ED = -W\sin i = -wl\cos i \sin i$;

" " " pressure at B $= BD = -H\cos i = -\dfrac{W\cos^2 i}{2\sin i}$; (3.)

" " " " " A $= AD = AE + ED$

$= DB + ED = H\cos i + W\sin i = \dfrac{W}{2}\left(\dfrac{1}{\sin i} + \sin i\right)$;

Transverse comp. of load $= -W\cos i = -wl\cos^2 i$; .... (4.)

" " sup. pres. $= H\sin i = \dfrac{W\cos i}{2\sin i}\cdot\sin i = \dfrac{W\cos i}{2}$ ......

The total longitudinal pressure at any point will evidently be the long. comp. of the weight above it + the long. comp. of the pressure at B.

$$\therefore p'A = H\cos i + wx\cos i\sin i \quad \therefore p' = \left(\frac{W\cos^2 i}{2\sin i} + wx\cos i\sin i\right)\frac{1}{A} \text{ ..... (5.)}$$

At the point A, $x = l$; and $wl\cos i = W$ and we have

$$p' = \frac{W}{2A}\left(\frac{1-\sin^2 i}{\sin i} + 2\sin i\right) = \frac{W}{2A}\left(\frac{1}{\sin i} + \sin i\right)$$

$p'' = \dfrac{M_1 m'h}{I}$. From eq.(8) p.285, §176, $M_1 = \frac{1}{12}Wl\cos i \quad \therefore p'' = \dfrac{Wl\cos i}{12}\cdot\dfrac{m'h}{I}$

$$\therefore f' = p' + p'' = \frac{W}{2A}\left(\frac{1}{\sin i} + \sin i\right) + \frac{Wl\cos i}{12}\cdot\frac{m'h}{I} \text{ .... (7A.)}$$

Let $q = \dfrac{I}{m'h^2 A}$ then $p' + p'' = \dfrac{W}{A}\left\{\frac{1}{2}\left(\frac{1}{\sin i} + \sin i\right) + \dfrac{l\cos i}{12qh}\right\}$ .... (7B.)

$$\therefore A = \frac{W}{f'}\left\{\frac{1}{2}\left(\frac{1}{\sin i} + \sin i\right) + \frac{l\cos i}{12qh}\right\} \text{ .... (8.)}$$

$q = \dfrac{I}{Am'h^2} = \dfrac{I}{Ahy_1} = \dfrac{n'bh^3}{m'h^2 A} = \dfrac{nbh}{A} = \dfrac{1}{6}$ for rectangle &c.

Table of values of q. I, II, III, IV, V, VI, present no difficulty.

VII $q = \dfrac{I}{Ahy_1}$ [see p.255] $= \dfrac{h'^2\left(\frac{A_2}{12} + \frac{A_1A_2}{4A}\right)}{Ah'\cdot\frac{h'}{2}\left(1 + \frac{A_1}{A}\right)} = \dfrac{\frac{A_2A_1 + A_2^2 + 3A_1A_2}{12A}}{\frac{(A_1+A_2)(2A_1+A_2)}{2A}}$

$$\therefore q = \frac{A_2(A_2 + 4A_1)}{6(2A_1 + A_2)(A_1 + A_2)}$$

VIII [see p.256]

$q = \dfrac{I}{Ahy_1} = \dfrac{h'^2\left\{\frac{A_2}{12} + \frac{A_2A_3 + A_1A_2 + 4A_1A_3}{4A}\right\}}{Ah'\cdot\frac{h'}{2}\,\frac{A_2 + 2A_3^*}{A}} = \dfrac{\frac{A_2A_1 + A_2^2 + A_2A_3 + 3A_2A_3 + 3A_1A_2 + 12A_1A_3}{12A}}{\frac{1}{2}(A_2 + 2A_3)}$

$$\therefore q = \frac{A_2(4A_1 + A_2 + 4A_3) + 12A_1A_3}{6(A_2 + 2A_3)(A_1 + A_2 + A_3)}$$

*From the figure p.75 Notes we see that y_1 is measured to the side A_3 instead of A_1 as it must be in the present case. Hence this is A_3 instead of A_1

IX In the above value of q make $A_3 = A_1$

p.295 §179 Cont.

$$q = \frac{A_2(8A_1+A_2)+12A_1^2}{6(A_2+2A_1)(A_2+2A_1)} = \frac{(A_2+2A_1)(A_2+6A_1)}{6(A_2+2A_1)(A_2+2A_1)}$$

$$= \frac{A_2+6A_1}{6(A_2+2A_1)} = \frac{1}{6}\left(1+\frac{4A_1}{A_2+2A_1}\right)$$

$$\text{Eq. (12) p. 273} \left\{ v_1' = \frac{n'''Wc^3}{Em'bh^3} \right\}$$

Deflection of AB $= v_s' = \dfrac{n''' W\cos i(\frac{1}{2}l)^3}{Em'bh^3}$; deflection of CD $= v_h' = \dfrac{n''' W(\frac{1}{2}l\cos i)^3}{Em'bh^3}$

$$\therefore v_s' : v_h' :: \cos i : \cos^3 i :: 1 : \cos^2 i \quad \text{.....(9.)}$$

The vertical component of $v_s' = v_s'\cos i$

$$\therefore v_s'\cos i : v_h' :: \cos i : \cos^2 i :: l : l\cos i \quad \text{.....(10.)}$$

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p.296 §179A

$$\text{Eq. (13) p. 273, § 169} \left\{ v_1 = \frac{n''f'c^2}{Em'h} \right\}$$

$$\therefore v = \frac{n''p_1c^2}{Em'h} \quad \therefore p_1 = \frac{Em'h}{n''c^2}v \quad \text{..... (1)}$$

The values of the factors given in the table on p.296 are taken directly from the articles referred to, with the exception of XVII which is taken from an article that was in the earlier editions [see 5th Ed. 1867, pp. 287, 288], occupying the space now occupied by what is given of the "Theorem of the Three Moments".

As that part of the article may have been omitted to make room for the new matter it is substantially inserted here; though the method is probably not so good in any case as the one remaining in the text on pp. 289, 290, 291.

The Continuous Girder.

"In determining the moment of resistance required in the parts of such a girder which are directly over the piers, the whole girder is supposed to be loaded with the travelling load as well as the fixed load and the method of § 176 (p. 283 eq. (1)) applies in order to find M_1

But the same method will not in this case give a sufficiently great value to M_0', the moment of resistance required at the middle of each span; for the greatest bending moment at that point is produced when one span is loaded with the travelling load as well as the fixed load, and the

p.296 adjoining spans loaded with the fixed load only.

§179 A. "Greater or less precision may be used in finding M′ according to circumstances. The following approximate method errs on the safe side and is best to employ when the fixed load is light compared with the travelling load.

"Calculate that part of M_0' which arises from the fixed load as for a girder fixed horizontal at the ends (Eq.(2) p.284 §176) and that part which arises from the travelling load as for a girder merely supported at the ends, but not fixed [Case IV p.246 §161]. In other words compute the bending moment which would be produced by the entire load, fixed and travelling, on a beam merely supported at the ends and subtract from the result the already computed moment of flexure M_1 at the points of support due to the fixed load alone."

w = intensity of fixed load,
w' = " " travelling load.

$$\S 176 \ \{ M_0' = M_0 - M_1 \} \quad \therefore \ M_0' = \frac{w+w'}{8} l^2 - M_1 \ \dots\dots \quad (1.)$$

$$\text{Eq.(8) p.285, } \S 176 \ \left\{ M_1 = \frac{wl^2}{12} \right\} \therefore \ M_0' = \frac{(w+w')l^2}{8} - \frac{wl^2}{12}$$

$$= \frac{w+3w'}{24} l^2 = \left(\frac{w}{3} + w' \right) \frac{l^2}{8} \ \dots\dots \quad (2.)$$

To find the deflection, calculate that produced by the total load in a beam merely supported at the ends, and subtract that due to the uniform moment M_1.

$$\text{Eq.(13) p.273, } \S 169 \ \left\{ v_1 = \frac{n'' f c^2}{E m' h} \right\} ; \ \text{Eq.(2A) p.252, } \S 162 \ \left\{ M_0 = \frac{fI}{y_1} \right\} ;$$

$$\therefore \ f = \frac{M_0 m' h}{I} ; \quad v_1' = \frac{n'' M_0 c^2}{EI} = \text{deflection due to total load};$$

$$v_1'' = \int_0^c i\,dx = -\int_0^c \int_0^x \frac{M_1}{EI} dx^2 = -\frac{M_1}{EI} \int_0^c \int_0^x dx^2 = -\frac{M_1 c^2}{2EI} = \text{deflection due to } M_1$$

$$\therefore \ v_1 = v_1' + v_1'' = \frac{n'' M_0 c^2}{EI} - \frac{M_1 c^2}{2EI}$$

For value of n'' see V p.274

$$\therefore \ v_1 = \frac{\frac{5}{12}(w+w')4c^4}{8} - \frac{w \cdot 4c^4}{24 EI} = \frac{c^4}{24EI}(5w + 5w' - 4w)$$

$$\therefore \ v_1 = (w + 5w') \frac{c^4}{24EI} \ \dots\dots \quad (3.)$$

Supposing the greatest stress to be in the middle the proof deflection $f' = \dfrac{M_0' m' h}{I}$ $\quad \therefore \dfrac{1}{I} = \dfrac{f'}{M_0' m' h} = \dfrac{f'}{(w+3w')\frac{c^2}{6} m' h}$

p.296 §179A Cont. Substituting this value of $\frac{1}{I}$ in equation (3.)

$$v_1 = \frac{w+5w'}{w+3w'} \cdot \frac{1}{4} \cdot \frac{f'c^2}{E m^2 h} \quad \text{-----------------} (4)$$

From which we have the factor given in the table p.296 XVII.

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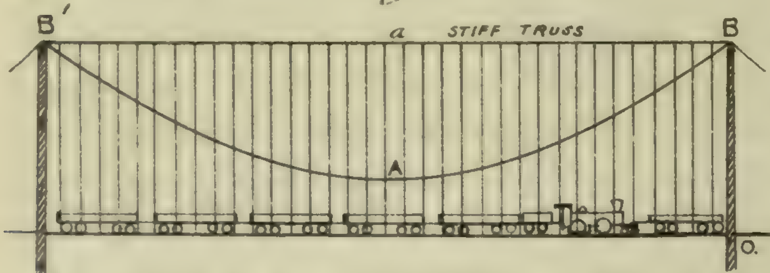
p.297 §180 PREFACE *Fig. 1*



To aid the student in comprehending §180 we will suppose, in fig. 1, that B'AB is a perfectly flexible chain and that B'aB is a perfectly stiff truss.

The chain has the form of a parabola and is therefore equilibrated for an uniformly distributed load (p.188, §.125). If in the figure the whole bridge was covered with cars of uniform weight, then the whole of the load would be carried by the chain, but if the bridge has a locomotive on it then the excess of the weight of the locomotive above that of a car, whose place it takes, will tend to pull the chain out of shape; but the chain is tied to the truss and, as soon as it moves the least bit, the excess of the load is supported by the truss; producing shearing stress and a bending moment in the truss.

Now in §180 w_1 corresponds to the intensity due to the weight of the bridge itself and of the cars, which in this case is constant and, $w - w_1$ to the excess of intensity due to the locomotive. The load producing tension on the chain is therefore $\int w_1 dx = H \frac{ds}{dx}$. And the load producing shearing and bending in the truss is $\int (w - w_1) dx$

$$\therefore F = F_0 - \int_0^x (w - w_1)\, dx$$

$$M = \int F dx = M_0 + F_0 x - \int_0^x \int_0^x (w - w_1)\, dx^2$$

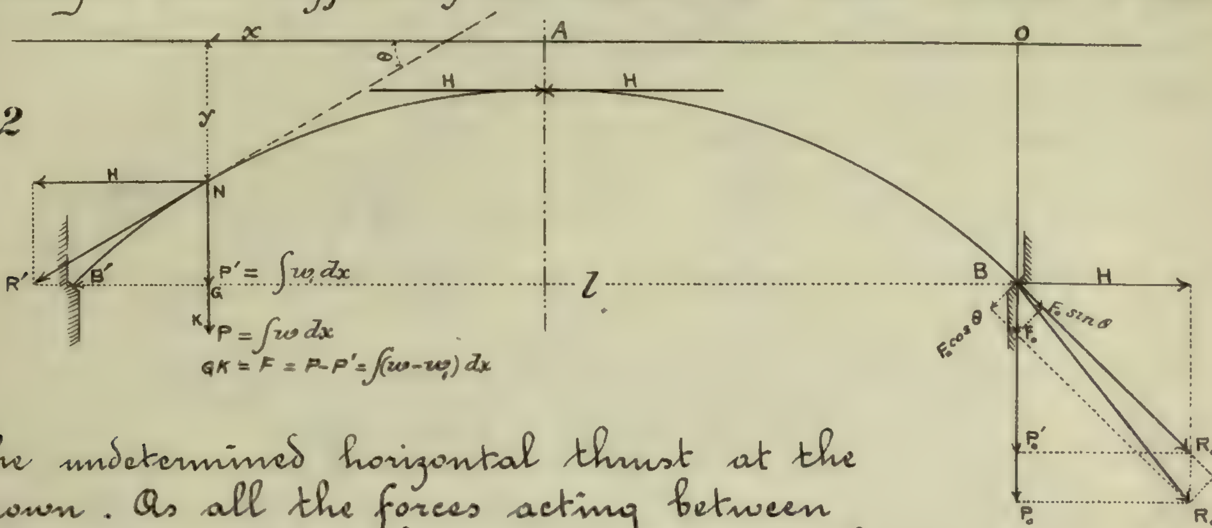
If there was a deficiency of load anywhere then we can suppose the truss pushes down, which would also give a shearing force and a bending moment but in the opposite direction from the case supposed above.

If we now suppose the truss removed and the chain made a stiff rib so as to itself resist the shearing and bending,

p.297 §.180 Cont. and then further suppose that it is inverted, we have the case discussed in PROBLEM IV, § 180 (nearly). Supposing it any equilibrated curve, and w_1 to correspond, we have the general case of § 180.

Strength and Stiffness of an Arched Rib under Vertical Loads.

Fig. 2



H = The undetermined horizontal thrust at the crown. As all the forces acting between the crown and the abutments are vertical the horizontal component of the thrust along the rib remains unchanged or equal to H

$P = \int w\,dx$ = The <u>Total</u> *Shearing* force on a vertical plane at any point. Which force can be found by considering that the vertical forces are all applied to a straight horizontal beam of the same length as the span; the x of the corresponding points in the two beams being the same. A part of this total shearing force, $P' = \int w_1 dx$, combines with H to give a simple thrust along the rib $= H \sec\theta$; while the remainder which in the text is called $F = \int (w - w_1)\,dx$, is the shearing force, on a vertical plane, which produces bending. This F is called by our author the "vertical component of shearing force". F may be decomposed into two components viz. $F\cos\theta$, acting at right angles to the rib, which is the true shearing force; and another $F\sin\theta$ acting in the direction of the rib and must be added to, or subtracted from, $H\sec\theta$, the thrust along the rib, to get the total thrust.

It is convenient however in most of the following discussion to keep this $F\sin\theta$ separate from the thrust which comes from H and P'; in fact not to decompose F at all.

This is brought out by the diagram of forces at B, fig 2, where H = the horizontal force at the crown, P_0 = total vertical force which depends on the distribution of

p.297 §180 cont. the load and is the same that the resistance of the abutment would be for a horizontal beam loaded in the same way.

The resultant of these two is R_o which, the direction being reversed, is the resistance of the abutment. We may decompose this R_o either into $R'_o = H \sec\theta$ and F_o, or into $R'_o + F\sin\theta$ and $F\cos\theta$; the former two being the forces mostly used in the text.

From fig. 2 we have

$$\tan\theta = \frac{dy}{dx} = \frac{P'}{H} = \frac{\int w_1 dx}{H} \;;\; \therefore \int w_1\, dx = H\frac{dy}{dx} \;;\; \therefore w_1 = H\frac{d^2y}{dx^2} \;;$$

$$\therefore (w - w_1)\, dx = \left(w - H\frac{d^2y}{dx^2}\right) dx \text{ .... } (1.)$$

$$F = \int (w - w_1)\, dx = F_o - \int_o^x (w - w_1)\, dx = F_o - \int_o^x w\, dx + \int_o^x H\frac{d^2y}{dx^2}\, dx$$

$$\therefore F = F_o - \int_o^x w\, dx + H\left(\frac{dy}{dx} - \frac{dy_o}{dx_o}\right) \text{ .... } (2.)$$

The bending moment at any section, as at x, is the same that it would be in a straight beam, equal in length to s, the rectified arc from B to the section x, and on which the shearing force is $F\cos\theta$.

$$\therefore M = \int F\cos\theta\, ds \;;\; \text{but } ds\cos\theta = dx \quad \therefore M = \int F\, dx\,,$$

which gives the same result as considering the shearing force, on vertical planes, which produces bending.

$$\therefore M = M_o + \int_o^x F\, dx = M_o + \int_o \left\{F_o - \int w\, dx + H\left(\frac{dy}{dx} - \frac{dy_o}{dx_o}\right)\right\} dx$$

$$\therefore M = M_o + \int_o^x F\, dx = M_o + F_o x - \int_o^x\!\!\int_o^x w\, dx^2 - H\left(y_o - y + x\frac{dy_o}{dx_o}\right) \text{ .... } (3.)$$

Fig. 3

The alteration of curvature produced in the arch, by that part of the load which tends to bend it, is $\frac{1}{r} = -\frac{M}{EI}$, the (−) being used for the reason given in the text.

The other part of the load $[\int w_1 dx]$ acting only in the direction of the curve produces no alteration in the curvature and may therefore be left out of consideration.

If the curve AB, fig. 3, be rectified into AB′, and at the points 1, 2, 3 &c ordinates be drawn equal to the <u>normal</u> deflections at the corresponding points 1, 2, 3 &c of the rib, we get the curve B′a; and $\frac{1}{r} = -\frac{M}{EI}$ represents the curvature of this curve of which r is the radius of curvature and i the inclination or slope.

p.298 §180 cont. The alteration of slope of the arch, or rib, is therefore

$$i = \int \frac{ds}{r} = i_0 - \int_0^s \frac{M}{EI}\, ds \;;\quad \text{but } ds = \sqrt{1+\frac{dy^2}{dx^2}}\cdot dx$$

$$\therefore i = i_0 - \int_0^x \frac{M}{EI}\sqrt{1+\frac{dy^2}{dx^2}}\cdot dx \quad \text{.................} \quad (4.)$$

Fig. 4

In fig. 4 let C and D be two points on the original curve whose distance apart is ds, and let c and d be the positions assumed by these points when the load is on the structure. The line joining D and C is tangent to the one curve and that joining c and d is tangent to the other, while the angle between them is i. Draw De parallel to cd then we see from the figure that

$$dv = Ce\cos\theta \;;\quad \text{but } Ce = i\,ds$$

$$\therefore dv = i\,ds\cos\theta \;;\quad \text{but } ds\cos\theta = dx \quad \therefore dv = i\,dx$$

since $v_0 = 0$ .... $\therefore v = \int_0^x i\,dx$ .... (5.)

The thrust along the rib at any point as N, fig 2, is $R' + F\sin\theta = H\sec\theta + F\sin\theta = H\frac{ds}{dx} + F\frac{dy}{ds}$. Intensity $= \dfrac{H\frac{ds}{dx} + F\frac{dy}{ds}}{A}$

p.226, § 149 $\left\{ \alpha = \frac{\Delta x}{x} = \frac{r}{E} \right\}$ $\therefore \dfrac{dt}{ds} = -\dfrac{H\frac{ds}{dx} + F\frac{dy}{ds}}{EA}$; but as the rib is usually very flat $\sin\theta$ is small, and F is generally small compared with H, so the term $F\frac{dy}{ds}$ may be neglected, which gives

$$\frac{dt}{ds} = -\frac{H\frac{ds}{dx}}{EA} \quad \text{........................} \quad (6.)$$

the minus sign being used because the length of the curve is decreased.

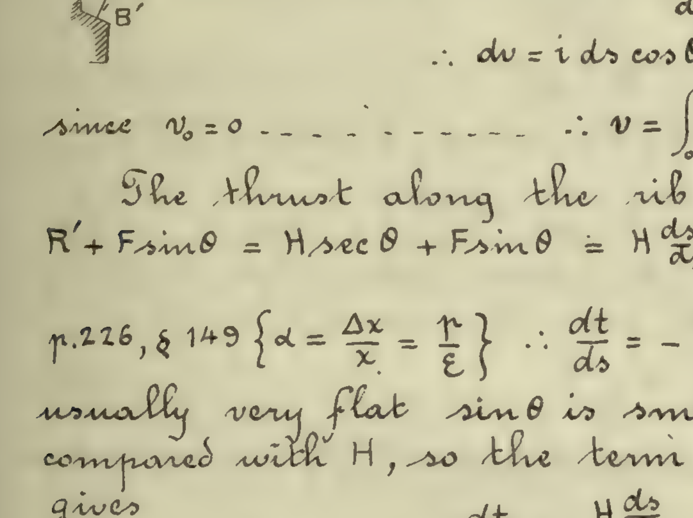


Fig. 5

From fig. 5

$OE = x$; $OF = x + dx$

$EC = y$; $FD = y + dy$

$Oe = x + u$; $Of = x + u + dx + du$

$ec = y + v$; $fd = y + v + dy + dv$

$CD = ds$; $cd = ds + dt$

From the two equations $ds^2 = dx^2 + dy^2$ and $(ds+dt)^2 = (dx+du)^2 + (dy+dv)^2$ } see fig. 4.

p.299 §180 Cont.

$$ds^2 + 2\,ds\,dt + dt^2 = dx^2 + 2\,dx\,du + du^2 + dy^2 + 2\,dy\,dv + dv^2$$

$$\therefore\ 2\,ds\,dt + dt^2 = 2\,dx\,du + du^2 + 2\,dy\,dv + dv^2$$

rejecting dt^2, du^2 and dv^2, because they are so small, we have

$$ds\,dt = dx\,du + dy\,dv$$

$$\therefore\ du = \frac{ds}{dx}dt - \frac{dy}{dx}dv \text{ .... } (7.)$$

But eq.(6) $\left\{ dt = -\frac{H\frac{ds^2}{dx}}{EA} \right\}$ also $\{dv = i\,dx\}$

$$\therefore\ du = -\frac{H}{EA}\cdot\frac{ds^3}{dx^2} - i\frac{dy}{dx}dx = -\frac{H}{EA}\frac{ds^3}{dx^3}dx - i\frac{dy}{dx}dx$$

since $\frac{ds^3}{dx^3} = \left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}$ $\therefore\ du = -\frac{H}{EA}\cdot\left(1+\frac{dy^2}{dx^2}\right)^{3/2}dx - i\frac{dy}{dx}dx$ .... (7A.)

$$u = -\int_0^x \left\{ \frac{H}{EA}\left(1+\frac{dy^2}{dx^2}\right)^{3/2} + i\frac{dy}{dx} \right\} dx \text{ .... } (8.)$$

The four undetermined constants are H, F_0, M_0 and i_0 but if the arched rib is fixed at the ends then $i_0 = 0$; and if it is not fixed at the ends then $M_0 = 0$ which reduces the number of unknown constants to three; which may be determined from equations $u_1 = 0$, or a given quantity* .... (9.)

$v_1 = 0$ (10.)

$i_1 = 0$, or $M_1 = 0$ .... (11.)&(11A.)

Where u_1, v_1, and i_1 or M_1 refer to the point of support B'; where $x = l$. See text.

p.73 eq.(d) Notes $\left\{ M = \frac{pI}{y} \right\}$ $\therefore\ p'' = \frac{Mm'h}{I}$ where p'' is the stress on the external fiber which comes from the bending moment, while that which comes from the direct thrust is

$$p' = \frac{H\frac{ds}{dx}}{A} \qquad \therefore\ p_1 = \frac{H\frac{ds}{dx}}{A} \pm \frac{Mm'h}{I} \text{ .... } (12.)$$

the thrust produced by F [which is $F\sin\theta = F\frac{dy}{ds}$] is not taken into account.

To find the vertical deviation of the line of resistance from the neutral curve at any point.

p.148 § 100 eq.(10) $\left\{ L = \frac{M}{R} \right\}$ $\therefore$ Vertical deviation $= \frac{M}{H}$ .... (13.)

$$\text{Normal deviation} = \frac{M}{H\sec\theta} = \frac{M}{H\frac{ds}{dx}} \text{ .... } (14.)$$

p.273 § 169 eq.(13) $\left\{ v_1 = \frac{n''f'c^2}{Em'h} \right\}$ $\therefore\ v = \frac{n''p''l^2}{4Em'h}$ $\therefore\ p'' = \frac{4Em'hv}{n''l^2}$

$$\therefore\ p_1 = \frac{H\frac{ds}{dx}}{A} \pm \frac{4Em'hv}{n''l^2} \text{ .... } (15.)$$

*See Problem III for $u_1 = aH$ = "a given quantity" a may be determined experimentally.

p.301 §180 Ex. PROBLEM II. *Rib of uniform stiffness.*–

For cross-sections of the same kind [p.252, §162] $A \propto bh$ and $I \propto bh^3$. If h is constant then $A \propto I \propto b$ but in this problem $b = b_1 \sec\theta$

$$\therefore A = A_1 \sec\theta\,; \quad I = I_1 \sec\theta \qquad \text{but } \sec\theta = \frac{ds}{dx} = \sqrt{1+\frac{dy^2}{dx^2}}$$

$$\therefore \quad A = A_1\sqrt{1+\frac{dy^2}{dx^2}}\;; \quad I = I_1\sqrt{1+\frac{dy^2}{dx^2}} \text{ ....} (16.)$$

A_1, I_1 and b_1 refer to the crown where $\sec\theta = 1$

Substituting for I its value in eq.(4) we have

$$i = \frac{dv}{dx} = i_0 - \frac{1}{EI_1}\int_0^x M\,dx \text{ ....} (4A.)$$

Which is the same value as the inclination of a horizontal beam with uniform cross-section I_1 and subjected to the bending moment M. Since $v = \int i\,dx$, v is also the same as for the supposed horizontal beam.

From eq.(6) $\left\{\frac{dt}{ds} = -\frac{H\frac{ds}{dx}}{EA}\right\}$ we have $\frac{dt}{ds} = -\frac{H}{EA_1}$ ....(6A.)

eq.(8) $\left\{u = -\int_0^x\left[\frac{H}{EA}\left(1+\frac{dy^2}{dx^2}\right)^{3/2} + i\,\frac{dy}{dx}\right]dx\right\}$ becomes, $u = -\frac{H}{EA_1}\int_0^x\left(1+\frac{dy^2}{dx^2}\right)dx - \int_0^x i\,\frac{dy}{dx}\,dx$ ...(8A.)

Substituting for i its value from eq.(4A)

$$u = -\frac{H}{EA_1}\int_0^x\left(1+\frac{dy^2}{dx^2}\right)dx - \int_0^x i_0\,dy + \frac{1}{EI_1}\int_0^x\frac{dy}{dx}\,dx\int_0^x M\,dx\;;$$

taking this between the limits, o and l, we have for the middle term $\quad -\int_0^l i_0\,dy = -i_0\int_0^l dy = -i_0(y_1 - y_0) = 0$ so this term drops out.

Substituting for I_1 its value $qm'h^2A_1$ the last term becomes $\frac{1}{Eqm'h^2A_1}\int_0^l\frac{dy}{dx}\int_0^x M\,dx^2$; hence we have for the value of the change in l

$$u_1 = \frac{1}{EA_1}\left\{-H\int_0^l\left(1+\frac{dy^2}{dx^2}\right)dx + \frac{1}{qm'h^2}\int_0^l\frac{dy}{dx}\int_0^x M\,dx^2\right\} \text{ ....} (18.)$$

eq.(12) $\left\{p_1 = \frac{H\frac{ds}{dx}}{A} \pm \frac{Mm'h}{I}\right\}$ becomes $\quad p_1 = \frac{H}{A_1} \pm \frac{Mm'h}{qm'h^2A} = \frac{1}{A_1}\left(H \pm \frac{M}{qh\sec\theta}\right)$

Rankine thinks the approximation sufficiently close to consider $\sec\theta = 1$ which gives

$$p_1 = \frac{1}{A_1}\left(H \pm \frac{M}{qh}\right) \text{ ....} (12A.)$$

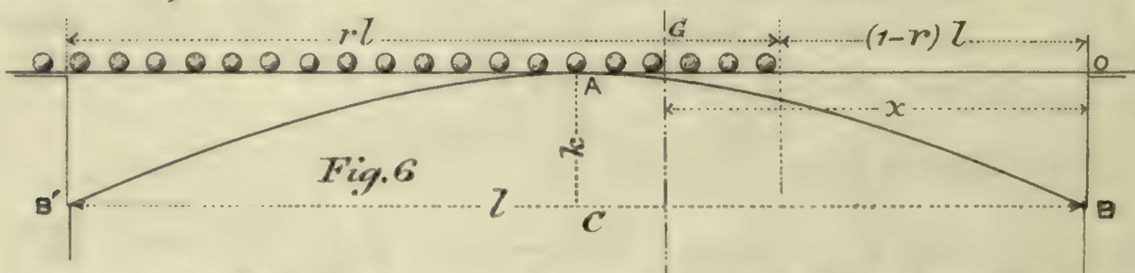
These formulæ may be applied to a flat segmental rib, with uniform cross-section, with but little error; because $\sec\theta = \frac{ds}{dx}$ will at no place differ much from unity.

p.302 §180 Ent. <u>Cor</u>. The change of temperature will change the horizontal span and a term expressing this should be introduced in the value of u in eq.(8A) thus if τ = greatest probable deviation of temperature from the assumed standard, and e = coefficient of expansion (see text p.539); then the horizontal expansion or contraction due to change of temperature is for the whole span $\int_0^l e\tau\,dx = e\tau l$; for any section $e\tau x$. Putting this in eq.(8A)

$$u = -\frac{H}{EA_1}\int_0^x \left(1 + \frac{dy^2}{dx^2}\right)dx \pm e\tau x - \int_0^x i\frac{dy}{dx}\,dx \quad \ldots\ldots\ldots (8A'.)$$

PROBLEM III, presents no difficulty, see text.

PROBLEM IV. *Parabolic Rib with Rolling Load; the Ends fixed in direction; the Abutments immovable.*



The most useful case is that of a rib of uniform stiffness; that is; h = constant; $b = b_1 \sec\theta$; $A = A_1 \sec\theta$; and $I = I_1 \sec\theta$.

If the ends are fixed $\quad i_0 = i_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots (19.)$

The equation to the rib with the origen at A is

$$x^2 = 2py$$

for the point B $\frac{l^2}{4} = 2pk \quad \therefore 2p = \frac{l^2}{4k}$; which gives $x^2 = \frac{l^2}{4k}y$; moving the origin to O we have

$$\left(\frac{l}{2}-x\right)^2 = \frac{l^2}{4k}y \quad \therefore y = \frac{4k}{l^2}\left(\frac{l}{2}-x\right)^2 \quad \ldots\ldots\ldots\ldots (20.)$$

From (20) by differentiation

$$\left.\begin{aligned} &\frac{dy}{dx} = -\frac{8k}{l^2}\left(\frac{l}{2}-x\right); \quad \frac{dy_0}{dx_0} = -\frac{4k}{l} \\ &1+\frac{dy^2}{dx^2} = 1 + \frac{64k^2}{l^4}\left(\frac{l}{2}-x\right)^2; \\ &\int_0^l \left(1+\frac{dy^2}{dx^2}\right)dx = l + \frac{16k^2}{3l}; \\ &\frac{d^2y}{dx^2} = \frac{8k}{l^2} \end{aligned}\right\} \quad (21.)$$

Also $\int_0^l i\frac{dy}{dx}dx = \int_0^l \frac{dy}{dx}.dv$; integrating this by parts

$\int u\,dz = uz - \int z\,du$; let $\left.\begin{matrix} u = \frac{dy}{dx}; & du = \frac{d^2y}{dx^2}dx; \\ dz = dv; & z = v; \end{matrix}\right\}$ and we have

p.304 §180 since $v_0 = v_1 = 0$ $\quad \int_0^l \frac{dy}{dx} \cdot dv = -\int_0^l \frac{d^2y}{dx^2} \cdot v\,dx = -\frac{8k}{l^2}\int_0^l v\,dx$ .... (22.)

Cor. This is simply proportional to the area of deflection, $\int_0^l v\,dx$

Let w_0 = intensity of permanent load; w = intensity of rolling load, applied as shown in fig. 6.

. Equations (2), (3), (4), (5) become as follows; where (A) refers to the unloaded division and (B) to the loaded.

Shearing Force.

eq. (2) is $\left\{ F = F_0 - \int_0^x w\,dx + H\left(\frac{dy}{dx} - \frac{dy_0}{dx_0}\right) \right\}$

$$\left.\begin{aligned}
&\therefore \text{(A.)} \quad F = F_0 - w_0 x + H\left[-\frac{8k}{l^2}\left(\frac{l}{2} - x\right) + \frac{4k}{l}\right] = F_0 + \left(\frac{8kH}{l^2} - w_0\right)x, \\
&\text{to get (B) substitute for } \int_0^x w\,dx \text{ in eq. (2) } \int_0^x w_0\,dx + \int_{(1-r)l}^x w\,dx \\
&\text{(B.)} \quad F = F_0 - w_0 x - wx + w(1-r)l + H\left[-\frac{8k}{l^2}\left(\frac{l}{2} - x\right) + \frac{4k}{l}\right] \\
&\qquad = F_0 + \left(\frac{8kH}{l^2} - w_0\right)x - w\left(x - (1-r)l\right).
\end{aligned}\right\} \text{.......(23.)}$$

Bending Moment. $\quad \left\{ M = \int F\,dx \right\} \therefore$

$$\left.\begin{aligned}
&\text{(A.)} \quad M = M_0 + F_0 x + \left(\frac{8kH}{l^2} - w_0\right)\frac{x^2}{2}; \\
&\text{(B.)} \quad M = M_0 + F_0 x + \left(\frac{8kH}{l^2} - w_0\right)\frac{x^2}{2} - \frac{w\left[x - (1-r)l\right]^2}{2}.
\end{aligned}\right\} \text{.........(24.)}$$

Alteration of Slope. Since the rib is of uniform stiffness we use eq. (4A.) $\left\{ i = i_0 - \frac{1}{EI_1}\int_0^x M\,dx \right\}$; but $i_0 = 0$ and $I_1 = qm'h^2A_1$ $\therefore$ substituting for M its value from eq. (24) and integrating,

$$\left.\begin{aligned}
&\text{(A.)} \quad i = \frac{1}{qm'h^2EA_1}\left\{-M_0 x - F_0\frac{x^2}{2} - \left(\frac{8kH}{l^2} - w_0\right)\frac{x^3}{6}\right\}; \\
&\text{(B.)} \quad i = \frac{1}{qm'h^2EA_1}\left\{-M_0 x - F_0\frac{x^2}{2} - \left(\frac{8kH}{l^2} - w_0\right)\frac{x^3}{6} + \frac{w}{6}\left[x - (1-r)l\right]^3\right\}.
\end{aligned}\right\} \quad (25.)$$

Deflection. $\quad \left\{ v = v_0 + \int_0^x i\,dx \right\}$; but $v_0 = 0 \quad \therefore$

$$\left.\begin{aligned}
&\text{(A.)} \quad v = \frac{1}{qm'h^2EA_1}\left\{-M_0\frac{x^2}{2} - F_0\frac{x^3}{6} - \left(\frac{8kH}{l^2} - w_0\right)\frac{x^4}{24}\right\} \\
&\text{(B.)} \quad v = \frac{1}{qm'h^2EA_1}\left\{-M_0\frac{x^2}{2} - F_0\frac{x^3}{6} - \left(\frac{8kH}{l^2} - w_0\right)\frac{x^4}{24} + \frac{w}{24}\left[x - (1-r)l\right]^4\right\}
\end{aligned}\right\} \quad (26.)$$

---

① The equations of condition are the following. $i_1 = 0$ and $\therefore$, from eq. (25), $\quad -M_0 - F_0\frac{l}{2} - \left(\frac{8kH}{l^2} - w_0\right)\frac{l^2}{6} + \frac{wr^3l^2}{6} = 0$ .... (27.)

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(2) $v_1 = 0$ ∴ from (B.)(26), (dividing by l^2)

$$-\frac{M_0}{2} - F_0\frac{l}{6} - \left(\frac{8kH}{l^2} - w_0\right)\frac{l^2}{24} + \frac{wr^4 l^2}{24} = 0 \quad \text{.................. (28)}$$

(3) $u_1 = 0$; on account of immovability of abutments;

eq.(8A.) is $\left\{u = -\frac{H}{EA_1}\int_0^x \left(1 + \frac{dy^2}{dx^2}\right)dx - \int_0^x i\,\frac{dy}{dx}\,dx\right\}$ when $x = l$ this reduces, see eq.(22), to $\quad -\frac{H}{EA_1}\int_0^l\left(1+\frac{dy^2}{dx^2}\right)dx + \frac{8k}{l^2}\int_0^l v\,dx = 0$

from eq.(21) & (26)

$$u_1 = 0 = -\frac{H}{EA_1}\left(l + \frac{16k^2}{3l}\right) + \frac{8k}{l^2}\cdot\frac{1}{qm'h^2EA_1}\left\{-\frac{M_0 l^3}{6} - \frac{F_0 l^4}{24} - \left(\frac{8kH}{l^2} - w_0\right)\frac{l^5}{120} + \frac{wr^5 l^5}{120}\right\}$$

multiplying by $\frac{qm'h^2EA_1}{8kl}$ we have

$$-\frac{M_0}{6} - \frac{F_0 l}{24} + \frac{w_0 l^2}{120} + \frac{wr^5 l^2}{120} - H\left\{\frac{k}{15} + q\frac{m'h^2}{8k}\left(1 + \frac{16k^2}{3l^2}\right)\right\} = 0 \quad \text{.............. (29)}$$

Eliminate M_0 and F_0 from the equations of condition to find H

Multiply (28) by 2 and subtract it from (27),

$$-F_0\frac{l}{6} - \left(\frac{8kH}{l^2} - w_0\right)\frac{l^2}{12} + \frac{wr^3l^2}{6} - \frac{wr^4l^2}{12} = 0\ ; \quad \text{.................. (a)}$$

multiply (29) by 3 and subtract it from (28),

$$-F_0\frac{l}{24} - \left(\frac{8kH}{l^2} - w_0\right)\frac{l^2}{24} + \frac{wr^4l^2}{24} - \frac{w_0l^2}{40} - \frac{wr^5l^2}{40} + H\left\{\frac{k}{5} + \frac{3qm'h^2}{8k}\left(1 + \frac{16k^2}{3l^2}\right)\right\} = 0 \quad \text{...(b)}$$

multiply (b) by 4 and subtract it from (a)

$$\left(\frac{8kH}{l^2} - w_0\right)\frac{l^2}{12} + \frac{wr^3l^2}{6} - \frac{wr^4l^2}{4} + \frac{wr^5l^2}{10} + \frac{w_0l^2}{10} - 4H\left[\frac{k}{5} + \frac{3qm'h^2}{8k}\left(1 + \frac{16k^2}{3l^2}\right)\right] = 0\ ;$$

collecting the terms containing H,

$$H\left[\tfrac{2}{3}k - \tfrac{4}{5}k - \frac{3qm'h^2}{2k}\left(1 + \frac{16k^2}{3l^2}\right)\right] = w_0\left(\frac{l^2}{12} - \frac{l^2}{10}\right) - wl^2\left(\frac{r^3}{6} - \frac{r^4}{4} + \frac{r^5}{10}\right);$$

$$\therefore\quad H\left[-\tfrac{2}{15}k - \tfrac{3}{2}\,q\frac{m'h^2}{k}\left(1 + \frac{16k^2}{3l^2}\right)\right] = -\frac{w_0l^2}{60} - wl^2\left(\frac{r^3}{6} - \frac{r^4}{4} + \frac{r^5}{10}\right);$$

multiplying through by $\frac{15}{2}$ and changing signs we get

$$H\left[k + \frac{45qm'h^2}{4k}\left(1 + \frac{16k^2}{3l^2}\right)\right] = l^2\left[\frac{w_0}{8} + w\left(\frac{15}{12}r^3 - \frac{15}{8}r^4 + \frac{15}{20}r^5\right)\right]$$

let $\frac{45qm'h^2}{4k^2}\left(1 + \frac{16k^2}{3l^2}\right) = B$; then .... (30.)

$$H(1+B)k = \frac{l^2}{8}\left[w_0 + w\left(10r^3 - 15r^4 + 6r^5\right)\right] \quad \therefore$$

$$H = \frac{l^2}{8(1+B)k}\left\{w_0 + w\left(10r^3 - 15r^4 + 6r^5\right)\right\} \quad \text{.............. (31.)}$$

Multiply (28) by 3 and subtract it from (27),

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$$\frac{M_o}{2} - \left(\frac{8kH}{l^2} - w_o\right)\frac{l^2}{24} + \frac{wr^3l^2}{6} - \frac{wr^4l^2}{8} = 0$$

$\therefore\ M_o = -\frac{w_o l^2}{12} - wl^2\left(\frac{r^3}{3} - \frac{r^4}{4}\right) + \frac{2}{3}kH$; and substituting for H its value,

$$M_o = -\frac{w_o l^2}{12} - wl^2\left(\frac{r^3}{3} - \frac{r^4}{4}\right) + \frac{1}{12}\cdot\frac{l^2}{1+B}\left[w_o + w\left(10r^3 - 15r^4 + 6r^5\right)\right];$$

$$\therefore\ -M_o = \frac{w_o l^2}{12}\cdot\frac{B}{1+B} + \frac{wl^2}{12}\left\{4r^3 - 3r^4 - \frac{10r^3 - 15r^4 + 6r^5}{1+B}\right\} \quad\text{............ (32.)}$$

Making $x = l$ in (B.) (24) we have $M_1 = M_o + F_o l + \left(\frac{8kH}{l^2} - w_o\right)\frac{l^2}{2} - \frac{wr^2l^2}{2}$; from eq. (a)

$$F_o l = wr^3l^2 - \frac{wr^4l^2}{2} - \left(\frac{8kH}{l^2} - w_o\right)\frac{l^2}{2} = \frac{wl^2}{12}\left(12r^3 - 6r^4\right) - \left(\frac{8kH}{l^2} - w_o\right)\frac{l^2}{2}$$

Substituting for M_o and $F_o l$ their values,

$$M_1 = -\frac{w_o l^2}{12}\cdot\frac{B}{1+B} - \frac{wl^2}{12}\left(4r^3 - 3r^4 - \frac{10r^3 - 15r^4 + 6r^5}{1+B}\right) + \frac{wl^2}{12}\left(12r^3 - 6r^4\right) - \left(\frac{8kH}{l^2} - w_o\right)\frac{l^2}{2}$$
$$+\left(\frac{8kH}{l^2} - w_o\right)\frac{l^2}{2} - \frac{wl^2}{12}\cdot 6r^2$$

$$\therefore\ -M_1 = \frac{w_o l^2}{12}\cdot\frac{B}{1+B} + \frac{wl^2}{12}\left(6r^2 - 8r^3 + 3r^4 - \frac{10r^3 - 15r^4 + 6r^5}{1+B}\right) \quad\text{............ (33.)}$$

Since the rib is fixed at the ends, and the moment of inertia is greater at the ends than any where else, it follows that the maximum bending moment occurs at the abutment, B′, next to the loaded end [see § 176, pp, 283, 284 and especially p.89 Notes 1st eleven lines].

Using eq. (12 A) $\left\{p_1 = \frac{1}{A_1}\left(H \pm \frac{M}{qh}\right)\right\}$ we see that the last term will have its greatest value when M is a max. therefore the max. value of p_1 occurs at the abutment on the side of the loaded end. This value then is

$$p_1 = \frac{1}{A_1}\left(H \pm \frac{M_1}{qh}\right).^*$$

Substituting for H and M_1 their values from eqs. (31) and (33)

$$p_1 = \frac{1}{A_1}\left\{\frac{l^2}{8(1+B)k}\left[w_o + w\left(10r^3 - 15r^4 + 6r^5\right)\right] + \frac{w_o l^2}{12qh}\cdot\frac{B}{1+B} + \frac{wl^2}{12qh}\left(6r^2 - 8r^3 + 3r^4 - \frac{10r^3 - 15r^4 + 6r^5}{1+B}\right)\right\}$$

$$\therefore\ p_1 = \frac{l^2}{8A_1}\left\{\frac{w_o}{1+B}\cdot\left(\frac{1}{k} + \frac{2B}{3qh}\right) + w\left[\frac{2}{3qh}\left(6r^2 - 8r^3 + 3r^4\right) - \left(\frac{2}{3qh} - \frac{1}{k}\right)\frac{10r^3 - 15r^4 + 6r^5}{1+B}\right]\right\} \quad\text{(34.)}$$

For tension let p_1' denote the stress, and q' the value of the factor q (the reason that q and q′ may differ is because the neutral axis may not be at ½ the depth of the rib), then

---

* *The exact formula is* $p_1 = \frac{1}{A_1}\left(H + \frac{F_1 \sin\theta}{\sec\theta} \pm \frac{M_1}{qh\sec\theta}\right)$; *see Notes on eqs. (12) and (12 A.).*

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Cont.

$$p_1' = \frac{1}{A_1}\left(\frac{M_1}{q'h} - H\right)$$

which by reducing as above gives

$$p_1' = \frac{l^2}{8A_1}\left\{\frac{w_0}{1+B}\left(\frac{2B}{3q'h} - \frac{1}{k}\right) + w\left(\frac{2}{3q'h}\left(6r^2-8r^3+3r^4\right) - \frac{10r^3-15r^4+6r^5}{1+B}\left(\frac{2}{3q'h}+\frac{1}{k}\right)\right)\right\} \text{......} (35.)$$

To find how much of the bridge must be loaded in order to give the greatest possible value to p_1 we apply to eq.(34) the test $\frac{dp_1}{dr}=0$, omitting the constant factors.

$$\frac{2}{3qh}\left[12r - 24r^2 + 12r^3\right] - \frac{30r^2-60r^3+30r^4}{1+B}\left(\frac{2}{3qh} - \frac{1}{k}\right) = 0 \text{ ....} (c.)$$

$$\therefore \frac{8}{qh} - \frac{16r}{qh} + \frac{8r^2}{qh} - \frac{30r}{1+B}\left(\frac{2}{3qh}-\frac{1}{k}\right) + \frac{60r^2}{1+B}\left(\frac{2}{3qh}-\frac{1}{k}\right) - \frac{30r^3}{1+B}\left(\frac{2}{3qh}-\frac{1}{k}\right) = 0$$

$$\therefore \frac{8}{qh} - r\left(\frac{16}{qh} + \frac{20}{(1+B)qh} - \frac{30}{(1+B)k}\right) + r^2\left(\frac{8}{qh} + \frac{40}{(1+B)qh} - \frac{60}{(1+B)k}\right) - r^3\left(\frac{20}{(1+B)qh} - \frac{30}{(1+B)k}\right) = 0$$

$$\therefore \frac{8(1+B)k}{qhk(1+B)} - r\left\{\frac{16(1+B)k+20k-30qh}{qhk(1+B)}\right\} + r^2\left\{\frac{8(1+B)k+40k-60qh}{qhk(1+B)}\right\} - r^3\left\{\frac{20k-30qh}{qhk(1+B)}\right\} = 0$$

$$\therefore r^3(20k-30qh) - r^2(48k+8Bk-60qh) + r(36k+16Bk-30qh) = 8(1+B)k$$

$$\therefore r^3 - r^2\left\{\frac{24k+4Bk-30qh}{10k-15qh}\right\} + r\left\{\frac{18k+8Bk-15qh}{10k-15qh}\right\} = \frac{4(1+B)k}{10k-15qh}$$

$$\therefore r^3 - r^2\left\{2 + \frac{4k+4Bk}{10k-15qh}\right\} + r\left\{1 + 2\left(\frac{4k+4Bk}{10k-15qh}\right)\right\} - \frac{4k+4Bk}{10k-15qh} = 0$$

This may be factored thus $(r^2-2r+1)\left(r - \frac{4k+4Bk}{10k-15qh}\right) = 0$; the roots are

$$r = +1\,,\ r = -1\,,\ r = \frac{4(1+B)k}{10k-15qh}.$$

From eq.(c)

$$\frac{d^2p_1}{dr^2} = \frac{24}{3qh}\left(1-4r+3r^2\right) - \left(\frac{60}{3qh} - \frac{30}{k}\right)\frac{2r-6r^2+4r^3}{1+B}$$

$$\frac{d^3p_1}{dr^3} = \frac{24}{3qh}\left(-4+6r\right) - \left(\frac{60}{3qh} - \frac{30}{k}\right)\frac{2-12r+12r^2}{1+B}$$

from which we see that $r=+1$ reduces $\frac{d^2p_1}{dr^2}$ to zero and does not so reduce $\frac{d^3p_1}{dr^3}$, therefore it does not correspond to a max.; $r=-1$ while it may mathematically correspond to a max. is physically absurd; and consequently the value of r which does correspond to a max. is

$$r_1 = \frac{4(1+B)k}{10k-15qh} = \frac{2}{5}\cdot\frac{1+B}{1-\frac{3qh}{2k}}$$

Similarly we find for tension

$$r_1' = \frac{2}{5}\cdot\frac{1+B}{1+\frac{3q'h}{2k}} \text{ ....} (36.)$$

p.307 §180 Cont. Substituting the above value of r_1 for r in that term in eq.(34) which involves r, we have.

$$\frac{2}{3qh}\left\{6\frac{(4(1+B)k)^2}{(10k-15qh)^2} - 8\frac{(4(1+B)k)^3}{(10k-15qh)^3} + 3\frac{(4(1+B)k)^4}{(10k-15qh)^4} - 10\frac{(4(1+B)k)^3}{(1+B)(10k-15qh)^3}\right.$$
$$\left. + \frac{15(4(1+B)k)^4}{(1+B)(10k-15qh)^4} - \frac{6(4(1+B)k)^5}{(1+B)(10k-15qh)^5} + \frac{1}{k}\cdot\frac{3qh}{2(1+B)}\left(\frac{10(4(1+B)k)^3}{(10k-15qh)^3} - \frac{15(4(1+B)k)^4}{(10k-15qh)^4} + \frac{6(4(1+B)k)^5}{(10k-15qh)^5}\right)\right\}$$

For brevity write $4(1+B)k = m$ and $10k - 15qh = n$; then the above becomes

$$\frac{2}{3qh}\left\{6\frac{m^2}{n^2} - 8\frac{m^3}{n^3} + \frac{3m^4}{n^4} - 10\frac{m^2 . 4k}{n^3} + 15\frac{m^3 . 4k}{n^4} - 6\frac{m^4 . 4k}{n^5}\right.$$
$$\left. + \frac{3qh . 5 . m^2 . 4}{n^3} - \frac{15}{2}\cdot\frac{3qh . m^3 . 4}{n^4} + 3\frac{3qh . m^4 . 4}{n^5}\right\}$$

$$= \frac{2}{3qh}\left\{6\frac{m^2}{n^2} - 8\frac{m^3}{n^3} + 3\frac{m^4}{n^4} - 4\frac{m^2}{n^3}(10k - 15qh) + 6\frac{m^3}{n^4}(10k - 15qh) - \frac{12}{5}\frac{m^4}{n^5}(10k - 15qh)\right\}$$

$$= \frac{2}{3qh}\left\{6\frac{m^2}{n^2} - 8\frac{m^3}{n^3} + 3\frac{m^4}{n^4} - 4\frac{m^2}{n^2} + 6\frac{m^3}{n^3} - \frac{12}{5}\frac{m^4}{n^4}\right\}$$

$= \frac{2}{3qh}\left\{2\frac{m^2}{n^2} - 2\frac{m^3}{n^3} + \frac{3}{5}\frac{m^4}{n^4}\right\}$; but $\frac{m}{n} = r_1$, hence for the max. value of p_1 we have

$$p_1 = \frac{l^2}{8A_1}\left\{\frac{w_0}{1+B}\left(\frac{2B}{3qh} + \frac{1}{k}\right) + \frac{2w}{3qh}\left(2r_1^2 - 2r_1^3 + \frac{3}{5}r_1^4\right)\right\} \quad (37.)$$

In a way precisely analogous, we find for max. tension p_1'

$$p_1' = \frac{l^2}{8A_1}\left\{\frac{w_0}{1+B}\left(\frac{2B}{3q'h} - \frac{1}{k}\right) + \frac{2w}{3q'h}\left(2r_1'^2 - 2r_1'^3 + \frac{3}{5}r_1'^4\right)\right\} \quad (38.)$$

Equation (37) serves to compute the proper sectional area when the depth and form of cross-section have been fixed

If eq.(38) gives a negative result there will be no tension. The value of F_0 may be obtained from eq.(a) by substituting the value of H therein; thus

$$-F_0 = \frac{l}{2}\left(\frac{8kH}{l^2} - w_0\right) - wr^3 l + \frac{wr^4 l}{2}$$

$$F_0 = \frac{l}{2}\left\{w_0 - \frac{1}{1+B}\left(w_0 + w(10r^3 - 15r^4 + 6r^5)\right) + 2wr^3 - wr^4\right\}$$

$$= \frac{l}{2}\left\{\frac{w_0 B}{1+B} + w\left(2r^3 - r^4 - \frac{10r^3 - 15r^4 + 6r^5}{1+B}\right)\right\} \quad (39.)$$

And this, together with M_0 and H, substituted in eq.(26) gives us the deflection.

If h, the depth of the arch, is small compared with k, its rise, we have $q\frac{h}{k}$ = a small fraction.
So in the case of B the first factor, $\frac{45qm'h^2}{4k^2}$, will be very small

p. 307 §180 Cont. and the second or $1+\frac{16k^2}{3l^2}$ will be less than 2 in most cases and when the arch is very flat it may be but a little over 1.

Then neglecting $\frac{qh}{k}$, $q'\frac{h}{k}$ and B, we have.

$$r_1 = r_1' = \tfrac{2}{5}\,, \text{------------------} (36\text{ A.})$$

$$p_1 = \frac{l^2}{8A_1}\left\{w_0\left(\frac{2B}{3qh}+\frac{1}{k}\right) + 0.138\,\frac{w}{qh}\right\}; \text{-------------} (37\text{ A.})$$

$$p_1' = \frac{l^2}{8A_1}\left\{w_0\left(\frac{2B}{3q'h}-\frac{1}{k}\right) + 0.138\,\frac{w}{q'h}\right\} \text{..............} (38\text{ A.})$$

From equation (36) we see that, when $(1+B)\div 1-\frac{3qh}{2k} \geq \frac{5}{2}$, the max. compression occurs when the load covers the entire bridge. Also that, when $(1+B)\div 1+\frac{q'h}{2k} \geq \frac{5}{2}$, the maximum tension occurs when the load covers the entire bridge

In this case $r_1 = 1 = r_1'$;

$$\therefore\quad H = \frac{l^2(w_0+w)}{8(1+B)k}\,; \text{----------------------} (31\text{ B.})$$

$$-M_0 = -M_1 = \frac{l^2}{12}\,\frac{(w_0+w)B}{1+B}\,; \text{-----------------} (33\text{ B.})$$

$$p_1 = \frac{l^2}{8A_1}\left\{\frac{w_0}{1+B}\left(\frac{2B}{3qh}+\frac{1}{k}\right) + 0.4\,\frac{w}{qh}\right\} \text{---------------} (37\text{ B})$$

$$p_1' = \frac{l^2}{8A_1}\left\{\frac{w_0}{1+B}\left(\frac{2B}{3q'h}-\frac{1}{k}\right) + 0.4\,\frac{w}{q'h}\right\} \text{..............} (38\text{ B.})$$

COROLLARY. To provide for the effects of temperature we have from the corollary to last problem, eq. (8A'.)

$$u = -\frac{H}{EA_1}\left(l+\frac{16k^2}{3l}\right) \pm \tau e l + \frac{8k}{l^2}\int_0^l v\,dx$$
$$= -\frac{H}{EA_1}\left(l+\frac{16k^2}{3l} \mp \frac{\tau e l\,EA_1}{H}\right) + \frac{8k}{l^2}\int_0^l v\,dx$$

Multiply by $q\frac{m'h^2EA_1}{8kl}$ and since the abutments are immovable

$$u_1 = 0 = -H\left\{\frac{k}{15} + q\frac{m'h^2}{8k}\left(1+\frac{16k^2}{3l^2} \mp \frac{\tau e E}{p_0}\right)\right\} + \text{etc.}$$

where p_0 is the mean intensity of pressure at the crown and since it depends on H its value will have to be found by approximation. In the above equation make

$$B = \frac{45qm'h^2}{4k^2}\left(1+\frac{16k^2}{3l^2} \mp \frac{\tau e E}{p_0}\right),$$ and use this value

p.308 §180 cont. of B in the formulæ of Prob. IV and the results will include the effects due to change of temperature. [see p.539 of the text]

Problem V.

If $u_1 = aH$, the expression from which eq. (29) is deduced will have the form

$$-\left(l + \frac{16k^2}{3l} + a\mathcal{E}A_1\right)\frac{H}{\mathcal{E}A_1} + 8\frac{k}{l^2}\int_0^l v\,dx = 0$$

and this can be reduced to a form similar to eq. (29). The value of H deduced from equations of condition, it will be found, may be brought into the same form as eq. (31) if we place

$$\frac{45\,q m' h^2}{4k^2}\left(1 + \frac{16k^2}{3l^2} + \frac{a\mathcal{E}A_1}{l}\right) = B.$$

To provide for change of temperature insert, as in the last problem, in the second factor of the value of B the term $\mp\frac{\tau e \mathcal{E}}{p_0}$.

Problem VI.

Take a rib similar to that in problem V but with ends not fixed. In this case $M_1 = M_0 = 0$ and i_0 is an undetermined constant. The action of the bending moment in this case is expressed as follows.

$$\left.\begin{aligned}
&(A.) \quad F = F_0 + \left(\frac{8kH}{l^2} - w_0\right)x \\
&(B.) \quad F = F_0 + \left(\frac{8kH}{l^2} - w_0\right)x - w[x - (1-r)l]
\end{aligned}\right\} \cdots\cdots (41.)$$

$$\left.\begin{aligned}
&(A.) \quad M = F_0 x + \left(\frac{8kH}{l^2} - w_0\right)\frac{x^2}{2} - \\
&(B.) \quad M = F_0 x + \left(\frac{8kH}{l^2} - w_0\right)\frac{x^2}{2} - w\frac{[x - (1-r)l]^2}{2}
\end{aligned}\right\} \cdots\cdots (42.)$$

$$\left.\begin{aligned}
&(A.) \quad i = i_0 - \frac{1}{q m' h^2 \mathcal{E}A_1}\left[F_0\frac{x^2}{2} + \left(\frac{8kH}{l^2} - w_0\right)\frac{x^3}{6}\right] \\
&(B.) \quad i = i_0 - \frac{1}{q m' h^2 \mathcal{E}A_1}\left[F_0\frac{x^2}{2} + \left(\frac{8kH}{l^2} - w_0\right)\frac{x^3}{6} - \frac{w}{6}(x - (1-r)l]^3\right.
\end{aligned}\right\} \cdots\cdots (43.)$$

$$\left.\begin{aligned}
&(A.) \quad v = i_0 x - \frac{1}{q m' h^2 \mathcal{E}A_1}\left\{F_0\frac{x^3}{6} + \left(\frac{8kH}{l^2} - w_0\right)\frac{x^4}{24}\right\} \\
&(B.) \quad v = i_0 x - \frac{1}{q m' h^2 \mathcal{E}A_1}\left\{F_0\frac{x^3}{6} + \left(\frac{8kH}{l^2} - w_0\right)\frac{x^4}{24} - \frac{w}{24}[x - (1-r)l]^4\right\}
\end{aligned}\right\} \cdots\cdots (44.)$$

If $u_1 = aH$, as in problem V, we have for the first equation of condition

p. 309
§180
Cont.

$$0 = -\left(1+\frac{16k^2}{3l^2}+\frac{a\,\mathcal{E}A_1}{l}\right)\frac{lH}{\mathcal{E}A_1}+\frac{8k}{l^2}\int_0^l v\,dx \text{ .... } (45.)$$

Substitute the value of v, from eq. 44, and integrate; then

$$0=-\frac{Hl}{\mathcal{E}A_1}\left(1+\frac{16k^2}{3l^2}+\frac{a\,\mathcal{E}A_1}{l}\right)+\frac{8k}{l^2}\left\{\frac{i_0 l^2}{2}-\frac{1}{qm'h^2\mathcal{E}A_1}\left[F_0\frac{l^4}{24}+\left(\frac{8kH}{l^2}-w_0\right)\frac{l^5}{120}-\frac{w}{120}r^5l^5\right]\right\}$$

Multiply by $q\dfrac{m'h^2\mathcal{E}A_1}{8k}$ and we have

$$(1)\quad q\frac{m'h^2\mathcal{E}A_1}{2}i_0-\frac{F_0l^2}{24}+\frac{w_0l^3}{120}+\frac{wr^5l^3}{120}-Hl\left\{\frac{k}{15}+q\frac{m'h^2}{8k}\left(1+\frac{16k^2}{3l^2}+\frac{a\,\mathcal{E}A_1}{l}\right)\right\}=0 \text{ ...... } (46.)$$

The next equation of condition $v_1=0$ is derived from (44); and multiplying it, by $q\dfrac{m'h^2\mathcal{E}A_1}{l}$, we have

$$(2)\quad qm'h^2\mathcal{E}A_1\,i_0-\frac{F_0l^2}{6}+\frac{w_0l^3}{24}+\frac{wr^4l^3}{24}-\frac{Hlk}{3}=0 \text{ .... } (47.)$$

And the third eq. of condition is $M_1=0$. Eq. (42) by dividing by l gives,

$$(3)\qquad F_0-\frac{w_0l}{2}-\frac{wr^2l}{2}+\frac{4kH}{l}=0 \text{ .... } (48.)$$

To eliminate i_0 from Eqs. (46) and (47), multiply the former by 2 and subtract it from the latter, and then divide by l^2, we have

$$-\frac{F_0}{12}+\frac{w_0l}{40}+\frac{wl}{120}(5r^4-2r^5)-\frac{Hk}{l}\left\{\frac{1}{5}-q\frac{m'h^2}{4k^2}\left(1+\frac{16k^2}{3l^2}+\frac{a\,\mathcal{E}A_1}{l}\right)\right\}=0 \text{ .... } (49.)$$

Multiply this by 12 and add it to eq. (48) and we get rid of F_0 $\therefore$

$$-\frac{w_0l}{5}-\frac{wl}{10}(5r^2-5r^4+2r^5)+\frac{Hk}{l}\left[\frac{8}{5}+\frac{3qm'h^2}{k^2}\left(1+\frac{16k^2}{3l^2}+\frac{a\,\mathcal{E}A_1}{l}\right)\right]=0 \text{ .... } (50.)$$

From this $$H=\frac{l}{k\;\frac{8}{5}\left[1+\frac{15}{8}q\frac{m'h^2}{k^2}\left(1+\frac{16k^2}{3l^2}+\frac{a\,\mathcal{E}A_1}{l}\right)\right]}\cdot\frac{l}{5}\left[w_0+\frac{w}{2}(5r^2-5r^4+2r^5)\right]$$

Placing $$\frac{15}{8}\cdot q\frac{m'h^2}{k^2}\left(1+\frac{16k^2}{3l^2}+\frac{a\,\mathcal{E}A_1}{l}\right)=C \text{ .... } (51.)$$

$$\therefore\qquad H=\frac{l^2}{8k(1+C)}\cdot\left[w_0+\frac{w}{2}(5r^2-5r^4+2r^5)\right] \text{ .... } (52.)$$

From Eq. (48) $F_0=\dfrac{w_0l}{2}+\dfrac{wr^2l}{2}-\dfrac{4kH}{l}$, substitute for H and reduce

$$F_0=\frac{l}{2}\left[\frac{w_0C}{1+C}+w\left(r^2-\frac{5r^2-5r^4+2r^5}{2(1+C)}\right)\right] \text{ .... } (53.)$$

From eq. (47) $qm'h^2\mathcal{E}A_1\,i_0=\dfrac{F_0l^2}{6}-\dfrac{w_0l^3}{24}-\dfrac{wr^4l^3}{24}+\dfrac{Hlk}{3}$; substituting for F_0 and H

$$qm'h^2\mathcal{E}A_1\,i_0=\frac{l^3}{24}\left[\frac{2w_0C}{1+C}+2w\left(r^2-\frac{5r^2-5r^4+2r^5}{2(1+C)}\right)\right]-\frac{l^3}{24}w_0-\frac{l^3}{24}wr^4+\frac{l^3}{24}\left[\frac{w_0}{1+C}+\frac{w}{2}\cdot\frac{5r^2-5r^4+2r^5}{1+C}\right]$$

$$=\frac{l^3}{24}\left[\frac{w_0C}{1+C}+w\left(2r^2-r^4-\frac{5r^2-5r^4+2r^5}{2(1+C)}\right)\right]$$

p.310 §180 $i_0 = \dfrac{l^3}{24 q m' h^2 E A_1}\left\{\dfrac{w_0 C}{1+C} + w\left(2r^2 - r^4 - \dfrac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right\}$ . . . .(54.)

Cont. The shearing force at the loaded end of rib (with sign reversed) is, from eq.(41.),

$P = -F_1 = -F_0 + w_0 l + wrl - \dfrac{8kH}{l}$; replacing F_0 by its value from eq.(48)

$P = \dfrac{w_0 l}{2} + \dfrac{wl}{2}(2r - r^2) - \dfrac{4kH}{l}$; replacing H by its value from Eq.(52)

$$P = \frac{l}{2}\left\{\frac{w_0 C}{1+C} + w\left(2r - r^2 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right\} \qquad (55.)$$

To avoid negative signs this is denoted by P

To find the maximum bending moment.

From eq.(42) $\dfrac{dM}{dx} = 0 = F_0 + x\left(\dfrac{8kH}{l^2} - w_0\right) - w(x - (1-r)l)$

$$\therefore x = \frac{F_0 + wl(1-r)}{w_0 + w - \frac{8kH}{l^2}} = \frac{-P + w_0 l + wrl - \frac{8kH}{l} + wl(1-r)}{w_0 + w - \frac{8kH}{l^2}}$$

Hence the distance from the loaded end is

$$l - x = \frac{P}{w_0 + w - \frac{8kH}{l^2}} \qquad (56.)$$

Substitute the value just found for x in eq.(42) and we find the max. bending moment. Thus putting $w_0 + w - \dfrac{8kH}{l^2} = s$ for convenience

$$M' = F_0\left(l - \frac{P}{s}\right) + \frac{1}{2}\left(\frac{8kH}{l^2} - w_0\right)\left(l - \frac{P}{s}\right)^2 - w\,\frac{\left(rl - \frac{P}{s}\right)^2}{2}$$

$$= F_0\left(l - \frac{P}{s}\right) + \frac{1}{2}\left(\frac{8kH}{l^2} - w_0\right)\left(l^2 - \frac{2Pl}{s} + \frac{P^2}{s^2}\right) - \frac{w}{2}\left(r^2 l^2 - \frac{2rlP}{s} + \frac{P^2}{s^2}\right)$$

$$= F_0\left(l - \frac{P}{s}\right) + \left(\frac{8kH}{l^2} - w_0\right)\left(\frac{l^2}{2} - \frac{Pl}{s}\right) - \frac{w}{2}\left(r^2 l^2 - \frac{2Prl}{s}\right) + \frac{1}{2}\left(\frac{8kH}{l^2} - w_0 - w\right)\frac{P^2}{s^2}$$

$$= F_0\left(l - \frac{P}{s}\right) + \left(\frac{8kH}{l^2} - w_0\right)\left(\frac{l^2}{2} - \frac{Pl}{s}\right) - \frac{w}{2}\left(r^2 l^2 - \frac{2Prl}{s}\right) - \frac{1}{2}\frac{P^2}{s};$$

now from eq.(48) $F_0 = \dfrac{w_0 l}{2} + \dfrac{wr^2 l}{2} - \dfrac{4kH}{l}$ $\therefore$

$$M' = \frac{w_0 l^2}{2} + \frac{wr^2 l^2}{2} - 4kH - \frac{w_0 lP}{2s} - \frac{wr^2 lP}{2s} + \frac{4kHP}{ls} + 4kH - \frac{w_0 l^2}{2} - \frac{8kHP}{ls} + \frac{w_0 Pl}{s} - \frac{wr^2 l^2}{2} + \frac{wrlP}{s} - \frac{1}{2}\frac{P^2}{s}$$

$$\therefore M' = \frac{w_0 lP}{2s} + \frac{2wrlP}{2s} - \frac{wr^2 lP}{2s} - \frac{8kHP}{2ls} - \frac{P^2}{2s}$$

$$= \frac{1}{2s}\left\{\left(w_0 l + 2wrl - wr^2 l - \frac{8kH}{l}\right)P - P^2\right\};$$

but from eq.(55) $w_0 l + 2wrl - wr^2 l - \dfrac{8kH}{l} = 2P$; hence

$$\left.\begin{aligned} M' &= \frac{1}{2s}(2P^2 - P^2) = \frac{P^2}{2\left(w_0 + w - \frac{8kH}{l^2}\right)} \\ \therefore M' &= \frac{P(l-x)}{2} \end{aligned}\right\} \qquad (57.)$$

p. 311 §180 Cont. As in problem IV the greatest stress will be

$$p_1 = \frac{1}{A_1}\left(H + \frac{M'}{qh}\right) \text{ - } (58.)$$

$$p_1 = \frac{1}{A_1}\left\{\frac{l^2}{8k(1+C)}\left[w_0 + \frac{w}{2}(5r^2 - 5r^4 + 2r^5)\right] + \frac{l^2}{4qh^2\left(w + w_0 - \frac{8kH}{l^2}\right)}\left[\frac{w_0 C}{1+C} + w\left(2r - r^2 - \frac{5r^2 - 5r^4 + 2r^5}{2(1+C)}\right)\right]^2\right\}$$

This expression still has H in the denominator while the numerator involves r^{10}. Hence when we differentiate to find the value of r, corresponding to the max. value of p_1, the fraction, the numerator of which involves r^{10} and the denominator r^5 we will obtain terms involving r^{14} in the equation $\frac{dp_1}{dr} = 0$; which equation is too complex for use. The value $r = \frac{1}{2}$, however gives a close approximation to the absolute maximum thrust, as has been found by trial.

Introducing this value into eq. (52) &c. we find

$$H = \frac{l^2}{8k(1+C)}\left(w_0 + \frac{w}{2}\right); \text{ - - - - - - - - - - - - - - - - - - } (52.A.)$$

$$F_0 = \frac{l}{2}\left[\left(w_0 + \frac{w}{4}\right)\frac{C}{1+C} - \frac{w}{4(1+C)}\right]; \text{ or } = \frac{l}{2}\left[\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C} - \frac{w}{4}\right], \ldots\ldots\ldots\ldots (53A.)$$

(by adding and subtracting r^2);

$$i_0 = \frac{l^3}{24qm'h^2EA_1}\left\{\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C} - \frac{w}{16}\right\}; \ldots\ldots\ldots\ldots\ldots\ldots (54A.)$$

$$P = \frac{l}{2}\left\{\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C} + \frac{w}{4}\right\}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (55A.)$$

by adding and subtracting $2r^2$ from the quantity in the brackets in the value of P (eq. 55);

$$l - x = \frac{\frac{l}{2}\left[\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C} + \frac{w}{4}\right]}{w_0 + w - \frac{w_0 + \frac{w}{2}}{1+C}} = \frac{\frac{l}{4}\left\{w + 4\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C}\right\}}{\frac{2\left\{\frac{w}{2} + (w_0 + w)C\right\}}{1+C}} = \frac{l}{4}\cdot\frac{w + 4\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C}}{w + 2\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C}} \ldots (56A.)$$

$$M' = \frac{P(l-x)}{2} = \frac{l^2}{64}\,\frac{\left[w + 4\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C}\right]^2}{w + 2\left(w_0 + \frac{w}{2}\right)\frac{C}{1+C}} \ldots\ldots\ldots\ldots\ldots (57A.)$$

For numerical example see text.

COR. I. If we suppose the arch hinged at the crown as
p. 541 well as at the abutments, which supposition is made in
§374 Case IV of Art. 374, p. 541, the formulæ reduce as follows.

In this case M at the crown = 0; we thus have a new equation of condition and can dispense with one of the other conditions. Let the moment at the crown be called M_c then from eq. (42)

p.541 [p.312, §180 cont.]

§374 Cont.

$$M_c = 0 = F_0\frac{l}{2} + \left(\frac{8kH}{l^2} - w_0\right)\frac{l^2}{8} - w\,\frac{\left(\frac{l}{2} - l + rl\right)^2}{2};$$

$$\therefore \frac{M_c}{l} = 0 = \frac{F_0}{2} - \frac{w_0 l}{8} + \frac{Hk}{l} - \frac{w}{2}\left(r^2 l - rl + \frac{l}{4}\right);$$

but from eq.(48) we have $\frac{M_1}{2l} = 0 = \frac{F_0}{2} - \frac{w_0 l}{4} - \frac{wr^2 l}{4} + \frac{2Hk}{l}$.

Placing these two equal to each other, and collecting terms, we find

$$H = \frac{l}{k}\left\{\frac{w_0 l}{8} - wl\left(\frac{r^2}{4} - \frac{r}{2} + \frac{1}{8}\right)\right\} = \frac{l^2}{k}\left\{\frac{w_0}{8} - w\left(\frac{r^2}{4} - \frac{r}{2} + \frac{1}{8}\right)\right\} \quad \text{..........(52 X.)}$$

$F_0 = \frac{w_0 l}{2} + \frac{wr^2 l}{2} - 4\frac{kH}{2}$, substitute for H its value,

$$F_0 = \frac{w_0 l}{2} + \frac{wr^2 l}{2} - \frac{w_0 l}{2} + wlr^2 - 2wlr + \tfrac{1}{2}wl;$$

$$\therefore F_0 = wl\left(\tfrac{3}{2}r^2 - 2r + \tfrac{1}{2}\right) \quad \text{..................(53 X.)}$$

So in eq.(55) $P = -F_1 = -F_0 + w_0 l + wrl - \frac{8kH}{l}$; substituting for H & F_0 their values and reducing, $P = wl\left(\frac{r^2}{2} - r + \frac{1}{2}\right)$ ....(55 X.)

From eq. 57 we have for the max. moment $M' = \frac{P^2}{2\left(w_0 + w - \frac{8kH}{l^2}\right)}$

$$\therefore M' = \frac{w^2 l^2\left(\frac{1}{2}r^2 - r + \frac{1}{2}\right)^2}{4w - 8wr + 4wr^2} \quad \text{..................(57 X.)}$$

When $r = \frac{1}{2}$ $\quad H = \frac{l^2}{8k}\left(w_0 + \frac{w}{2}\right)$; .... (3.)

$$P = \tfrac{1}{8}wl\ ;\ F_0 = -\tfrac{1}{8}wl\ ;\ M' = \frac{wl^2}{64} \quad \text{.................. (4.)}$$

$$p_1 = \frac{1}{A_1}\cdot\left(\frac{M'}{qh} + H\right) = \frac{1}{A_1}\left[\frac{l^2 w}{64}\cdot\frac{1}{qh} + \frac{l^2}{8k}\left(w_0 + \frac{w}{2}\right)\right] = \frac{l^2}{8A_1}\left[\frac{w}{8qh} + \frac{1}{k}\left(w_0 + \frac{w}{2}\right)\right] \quad \text{.......... (5.)}$$

Also when $q' = q$ $\quad p_1' = \frac{1}{A_1}\left(\frac{M'}{qh} - H\right) = \frac{l^2}{8A_1}\left[\frac{w}{8qh} - \frac{1}{k}\left(w_0 + \frac{w}{2}\right)\right]$ .... (6.)

$$\therefore \frac{p_1 - p_1'}{p_1 + p_1'} = \frac{\frac{l^2}{8A_1}\left[\frac{2}{k}\left(w + \frac{w}{2}\right)\right]}{\frac{l^2}{8A_1}\cdot\frac{w}{4qh}} = \frac{8}{k}\cdot\frac{w_0 + \frac{w}{2}}{\frac{w}{qh}} = \frac{qh}{k}\cdot\frac{8w_0 + 4w}{w}$$

$$\therefore \frac{qh}{k} = \frac{w}{8w_0 + 4w}\cdot\frac{p_1 - p_1'}{p_1 + p_1'} \quad \text{.................. (7.)}$$

Making $r = 1$ in eq.(52 X.) we have $H_1 = \frac{l^2}{8k}(w_0 + w)$ .... (8.)
[where H_1 is the max. value of H]

p.312 §180 Cont. By comparing equations (52 X.), (53 X.), (55 X.) & (57 X.) with the values of H, F_0, P & M′ as given in equations (52 A), (53 A.), (55 A.) & (57 A.) we see that the latter will reduce to the former

by making $C=0$.

p.312 §180 Ent. To compare the general values of H &c., obtained by using the equation $M_c=0$, with those obtained in (52) &c, by using $u_1=0$.

If in eqs (52.) and (52 X.) we make $C=0$, these two equations become identical only when

$$\frac{1}{8}wr^5 - \frac{5}{16}wr^4 + \frac{5}{16}wr^2 = -\left(\frac{wr^2}{4} - \frac{wr}{2} + \frac{w}{8}\right);$$

or when

$$\frac{1}{8}r^5 - \frac{5}{16}r^4 + \frac{9}{16}r^2 - \frac{r}{2} + \frac{1}{8} = 0;$$

or, we may write, $2r^5 - 5r^4 + 9r^2 - 8r + 2 = 0 = (x-1)(x-1)(x-\frac{1}{2})(x+\sqrt{2})(x-\sqrt{2})$

Hence the value $C=0$ only renders the values of H &c identical when $r=1$, or $r=\frac{1}{2}$, or $r=\pm\sqrt{2}$.

If we suppose the example given on p.312 of the text to be hinged at the crown we find,

$$H = 1.50\, lw;$$

$$p_1 = \frac{1}{A_1}\left[\frac{120wl}{64} + 1.50\, wl\right];$$

$$= \frac{wl}{A_1}\left[3.375\right].$$

When the rib is continuous at the crown as in the example on p. 312, $p_1 = \frac{wl}{A_1}(3.51)$. Hence the gain, in hinging at the crown, is in this case about 4%.

Perhaps the greatest advantage in having an arch hinged at the crown is that there are no temperature strains.

COR. 2. The effects of change of temperature can be introduced into the equations of <u>problem</u> VI in a manner similar to that used in <u>problem</u> IV.

PROBLEM VII. *To find the greatest deflection of an arched rib.*

Evidently the greatest value of v must occur when $i=0$, that is, when the alteration of slope having gone through all its positive values becomes zero, since $v=\int i\,dx$.

Find the value of x in eq. (25), when $i=0$, and substitute these values, or the one of them which will be evidently applicable, in eq. (26); Then placing $\frac{dv}{dr}=0$, find the value of r which will render v a max. This process gives equations of extreme intricacy and involving r to a high power; hence they are hardly available

p.312 §180 Ext. for use. But by making suppositions on them, which will include the ordinary practical cases, it has been determined that the absolute maximum deflection occurs at the middle of the rib and when the passing load extends over the entire length. Hence when v is a max., $r = 1$ and $x = \frac{l}{2}$.

Using these values of r and x, eq.(31) becomes

$$H = \frac{l^2(w+w_0)}{8k(1+B)}.$$

Eq.(39) becomes $$F_0 = \frac{l(w+w_0)B}{2(1+B)}$$

Eqs.(32) and (33) become $$-M_0 = -M_1 = \frac{l^2(w+w_0)B}{12(1+B)}$$

These values with those of r and x substituted in eq.(B)(26) give

$$v = \frac{1}{qm'h^2EA_1}\left\{\frac{l^2(w+w_0)B}{12(1+B)}\cdot\frac{l^2}{8} - \frac{l(w+w_0)B}{2(1+B)}\cdot\frac{l^3}{48} - \frac{w-Bw_0}{1+B}\cdot\frac{l^4}{384} + \frac{wl^4}{384}\right\} \qquad (61)$$

$$= \frac{1}{qm'h^2EA_1}\left\{\frac{l^2(w+w_0)B}{12(1+B)}\cdot\frac{l^2}{8} - \frac{l(w+w_0)B}{2(1+B)}\cdot\frac{l^3}{48} - \left(w - \frac{(w+w_0)B}{1+B}\cdot\frac{l^4}{384}\right) + \frac{wl^4}{384}\right\}$$

$$= \frac{1}{qm'h^2EA_1}\left\{\frac{l^4(w+w_0)B}{96(1+B)} - \frac{l^4(w+w_0)B}{96(1+B)} + \frac{l^4(w+w_0)B}{384(1+B)}\right\}$$

$$\therefore v = \frac{l^4(w+w_0)B}{384\,qm'h^2EA_1(1+B)}$$

If $c = \frac{l}{2}$ (as in §169 &c) and $l(w+w_0) = W$, then the above may be written

$$v = \frac{Wc^3}{48EI_1}\cdot\frac{B}{1+B}.$$

So too eq.(52) becomes $$H = \frac{l^2(w+w_0)}{8k(1+C)};$$

Eq.(53) becomes $$F_0 = \frac{l(w+w_0)C}{2(1+C)};$$

Eq.(54) " $$i_0 = \frac{l^3(w+w_0)C}{24\,qm'h^2EA_1(1+C)}. \qquad (62)$$

Substitute, in eq.(B)(44), and reducing in precisely the same way as in finding the value of v above.

$$v = \frac{5l^4(w+w_0)C}{384\,qm'h^2EA_1(1+C)}.$$

Placing $l = 2c$, and $W = (w+w_0)l$, this becomes

$$v = \frac{5c^3W}{48EI_1}\cdot\frac{C}{1+C}.$$

Now eq.(12) p.273 gives when we substitute for n''' from Case V

p.313 page 274 $$v_1 = \frac{5}{48} \cdot \frac{Wc^3}{\mathcal{E} I_1}.$$

§180 Cont. Hence if W, c and I_1 be the same in this as in eq.(62)

$$v : v_1 :: \frac{C}{1+C} : 1 :: C : 1+C$$

When the straight beam is fixed at the ends, eq.(4) p.284 shows that the deflection is diminished in the ratio $n'' : (n'' - \frac{m''}{2})$ which in this case is $\frac{5}{12} : (\frac{5}{12} - \frac{4}{12}) :: 5 : 1$

Hence in such a beam $v_1 = \frac{1}{48} \cdot \frac{Wc^3}{\mathcal{E} I}$ and this compared with

eq.(61) gives $$v : v_1 :: \frac{B}{1+B} : 1 :: B : 1+B$$

PROBLEM VIII.

The value of I_1 must include the moment of inertia of the straight beam, because the straight beam helps to resist the bending moment, and I_1 occurs in formulæ relating to bending while A_1 does not.

The area A_1 must be that of the rib only, because the rib alone resists the direct thrust in the direction of the rib and A_1 occurs in formulæ for finding the intensity of this thrust while I_1 does not.

p.318 §183

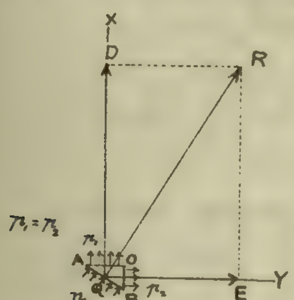
Theory of the Stability and Pressure of Loose Earth

In order to explain this article it will be necessary to discuss further the Ellipse of Stress

Rankine's method of determining the stress on any plane is based on the following two problems both of which are derived from the Ellipse of Stress and stated on p.40 Notes; but may be proven independently. (see A.M. §110, 111)

EQUAL PRINCIPAL STRESSES.

I Same Kind, Fluid Pressure. $p_1 = p_2$

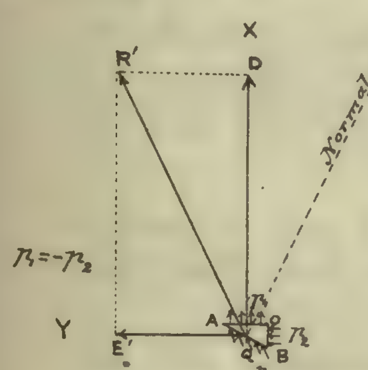


Let it be required to find the direction and intensity of the stress on any plane as AB. Consider the conditions of equilibrium of the prism AOB whose length perpendicular to the plane of the paper is unity. Then the total pressure on OB = p_2OB and is represented by the line QE; the total pressure on OA = p_1OA and is represented by QD. The resultant of these two QR must represent in direction and magnitude the total stress on AB.

$$\therefore \text{Intensity on AB} = \frac{QR}{AB} = \frac{p_1\sqrt{\overline{OB}^2+\overline{OA}^2}}{\sqrt{\overline{OB}^2+\overline{OA}^2}} = p_1 = p_2 = p_r$$

It may be seen from the figure that QR is perpendicular to AB Therefore the intensity is the same on all planes and the direction normal which is Fluid Pressure.

II Different Kind, one tension the other compression, $p_1 = -p_2$.



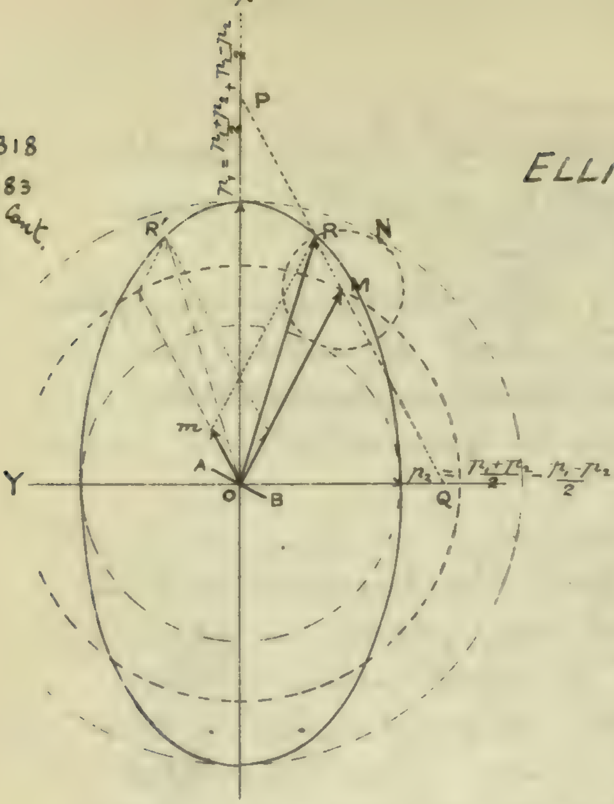
By analogy to the first problem and by inspection of the figures it is evident that the resultant stress will be of the same intensity as the principal stresses and that the angle which its direction makes with the normal to AB is bisected by the principal axes.

$$\text{That is } \frac{QR}{AB} = \frac{p_1\sqrt{\overline{OB}^2+\overline{OA}^2}}{\sqrt{\overline{OB}^2+\overline{OA}^2}} = p_1 = p_r \text{ ;}$$

and the angle NQD = R'QD.

From these two problems our author derives the Ellipse of stress as will be now shown.

p. 318
§ 183
Ent.



ELLIPSE OF STRESS.

When p_1 and p_2 are unequal each may be broken into two components.

$$p_1 = \frac{p_1+p_2}{2} + \frac{p_1-p_2}{2}$$

$$p_2 = \frac{p_1+p_2}{2} - \frac{p_1-p_2}{2} \;;$$

the first pair, being equal and of the same kind, their resultant is laid off on the normal and is $= \frac{p_1+p_2}{2} = OM$; the second pair, being equal but of different kind, their resultant is $Om = \frac{p_1-p_2}{2}$ and laid off on a line making the same angle with OX that the normal does, but on the opposite side. OR, the resultant of OM and Om, represents both the direction and intensity of the total stress on the plane AB. Hence Rankine's rule for finding the stress on any plane, when the principal stresses p_1 and p_2 are given, is to lay off on the normal the distance $OM = \frac{p_1+p_2}{2}$ then on a line making the same angle with the principal axes that the normal does, but in the opposite direction, lay off $MR = \frac{p_1-p_2}{2}$; then $OR = p$ is the required stress.

That the locus of the point R is an Ellipse is shown on p. 37 Notes.

If the essential sign of p_2 were negative that is if p_1 and p_2 were different in kind then OR′ would represent the direction and intensity of the stress on AB. Since in the pressure of earth the stresses are always of the same kind we will at present confine our discussion to that case.

The locus of M is a circle whose centre is O and whose radius is $\frac{p_1+p_2}{2}$ and the locus of R is a circle which has the moving centre M and the radius $\frac{p_1-p_2}{2}$.

The *ELLIPSE OF STRESS* may therefore be generated by the simultaneous revolutions of the point M about O and of the point R about M, the angular motion of the point R (measured from the radius OM) being double that of M and in the opposite direction. The angular motion, from a fixed direction, of R is the same as that of M, but one is right handed and the other left handed rotation.

p 318 §183 Cont. The greatest obliquity of stress on any plane evidently occurs when MR is perpendicular to OR. Let $\hat{nr}$ be this angle then

$$\text{Max. } \hat{nr} = \text{arc.} \sin \frac{MR}{OM} = \text{arc.} \sin \frac{p_1 - p_2}{p_1 + p_2}$$

$$\text{Max. } \hat{nr} \leqq \varphi \quad \therefore \quad \sin \varphi \geqq \sin \hat{nr} \quad \therefore$$

$$\frac{MR}{OM} = \frac{p_1 - p_2}{p_1 + p_2} \leqq \sin \varphi \quad \text{-------------} \quad (1)$$

$$\therefore p_1 - p_2 \leqq p_1 \sin \varphi + p_2 \sin \varphi \quad \therefore \quad p_1 - p_1 \sin \varphi \leqq p_2 + p_2 \sin \varphi$$

$$\therefore p_1 (1 - \sin \varphi) \leqq p_2 (1 + \sin \varphi) \quad \therefore \quad \frac{p_2}{p_1} \geqq \frac{1 - \sin \varphi}{1 + \sin \varphi} \quad \text{-----------} \quad (1A)$$

When the Ellipse of Stress is given we can find the plane on which the obliquity is a max. by taking the distance

$$OR = \sqrt{(\overline{OM}^2 - \overline{MR}^2)}$$

As OR may be measured on either side of OX we will have two conjugate stresses each of the intensity $\frac{1}{2}\sqrt{(p_1 + p_2)^2 - (p_1 - p_2)^2}$ each acting on the plane of the other; consequently these two planes are those on which the pressure is most oblique. Let the angle MOY $= \hat{ny}$ then

$$\hat{ny} = \frac{RMO}{2}$$

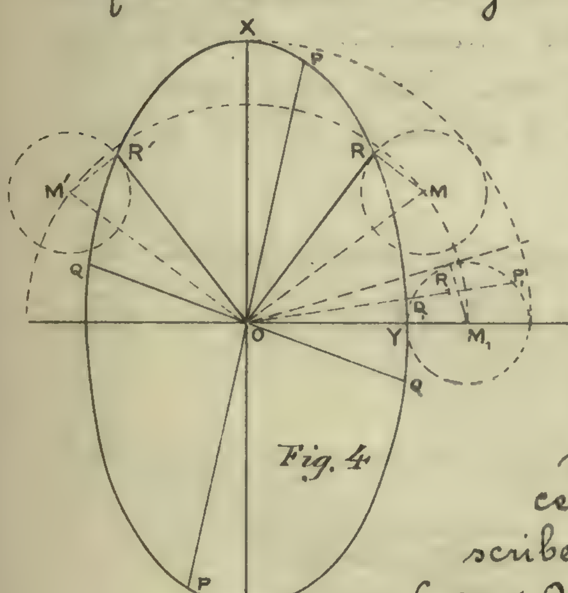


Fig. 4

In the same way the plane on which the obliquity is any given angle, less than max. $\hat{nr}$, may be located by means of the Ellipse of Stress; there being always two planes on which the obliquity is the same.

It is evident that if we take a distance $OM_1 = \frac{1}{2}(p_1 + p_2)$ and with M_1 as a centre and a radius $= \frac{1}{2}(p_1 - p_2)$ we describe a circle; then by drawing lines from O to the different points of this circle, the intensity and obliquity of every stress may be represented and knowing these two things the position of the plane on which the stress acts may be determined, as OR & OR' were in fig. 4.

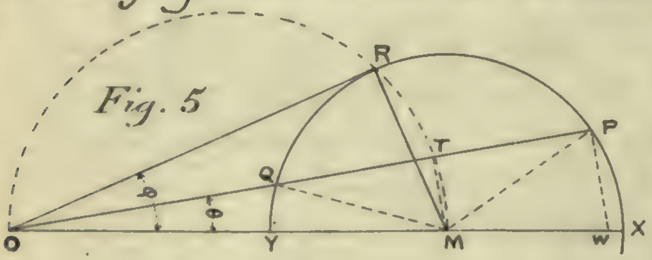
The important principle stated by our author in

p 319 §183 Cont. his Applied Mechanics, p. 215, § 196, must be constantly borne in mind in discussing problems relating to earth pressure.

PRINCIPLE OF LEAST RESISTANCE. *Theorem.* "If the forces which balance each other in or upon a given body or structure be destinguished into two systems, called respectively *active* and *passive*, which stand to each other in the relation of cause and effect, then will the passive forces be the least which are capable of balancing the active forces consistently with the physical condition of the body or structure."

"For the passive forces being caused by the active forces will not increase after the active forces have been balanced by them; and will therefore not increase beyond the least amount capable of balancing the active forces Q.E.D."

II *Conjugate Pressures in Earth.*



The accompanying figure, constructed as in the text, is obviously similar to a part of Fig. 4.

When the relation of the conjugate stresses is such that the maximum obliquity exists any where then the ratio of any two conjugate stresses p' and p whose obliquity is θ is that of OQ to OP

$$\therefore \frac{p'}{p} = \frac{OQ}{OP}$$

Also OM and MR(= MQ = MP) bear the ratio to each other of $\frac{1}{2}(p_1+p_2) \div \frac{1}{2}(p_1-p_2)$. It is evident that $\frac{p'}{p}$ cannot be less than $\frac{OQ}{OP}$ nor greater than $\frac{OP}{OQ}$ ——

To find the value of $\frac{OP}{OQ}$; $\quad \overline{OR}^2 = OP \times OQ = (OT+TP)(OT-TP) = \overline{OT}^2 - \overline{TP}^2$

$$\therefore TP = TQ = \sqrt{\overline{OT}^2 - \overline{OR}^2} = OM\sqrt{\cos^2\theta - \cos^2\varphi}$$

Hence $\quad OP = OT + TP = OM\left[\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}\right]$

$$OQ = OT - TQ = OM\left[\cos\theta - \sqrt{\cos^2\theta - \cos^2\varphi}\right]$$

$$\therefore \frac{p'}{p} \text{ cannot be greater than } \frac{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}} \quad \text{-------- (2.)}$$

$$\text{nor less than } \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}} \quad \text{------------------ (2A)}$$

p 320
§183

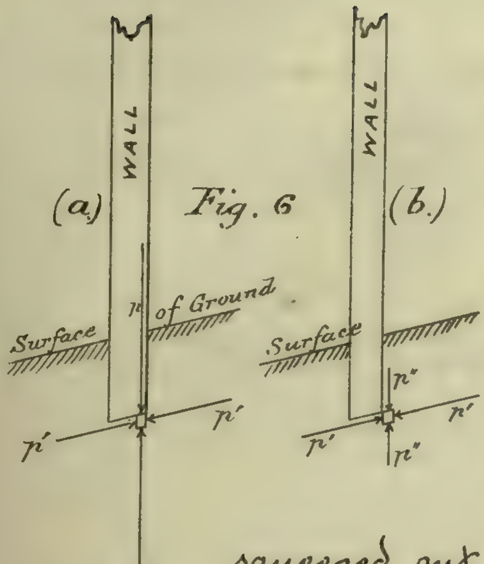


Fig. 6

As an example under this head suppose it is required to know the depth to which a foundation must be dug.

A vertical wall causes a pressure on the foundation of intensity p, and this will cause a conjugate pressure, parallel to the surface of the ground, along the steepest slope, whose intensity is $p' = p\,\frac{OQ}{OP}$ see Fig. 6 (a).

In order to balance the pressure p' and preserve the earth from being squeezed out from under the wall a vertical pressure of intensity $p'' = p'\,\frac{OQ}{OP}$ is required. This pressure must be produced by the weight of the earth above the bottom of the wall. If x be the depth of the foundation in inches and w the weight of a cu. in. of earth then

$$x = \frac{p''}{w\cos\theta}$$

III

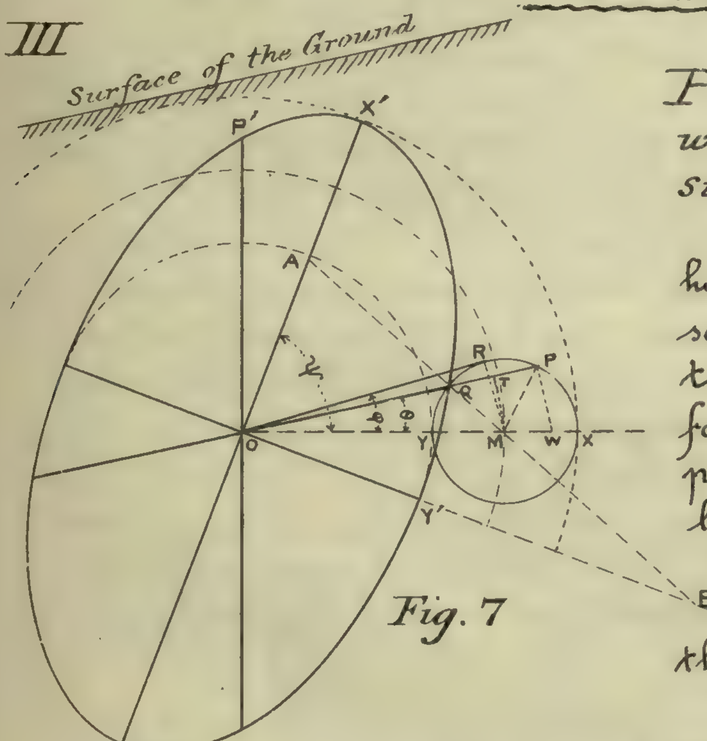


Fig. 7

SECTION ON STEEPEST SLOPE

Pressure of a mass of earth with an unlimited plane upper surface.

The condition of stress here is represented by an ellipsoid. The directions of the three conjugate stresses are, for reasons given in the text, p vertical, p' parallel to the line of greatest slope, and p'' at right angles to the plane of the other two and therefore horizontal.

We first determine the section of this ellipsoid, in the plane $p\,p'$, which is represented in Fig. 7

From eq. 2A, we have, $\frac{p'}{p} = \frac{p'}{w x \cos\theta} = \frac{OQ}{OP} = \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}$.

p. 320 §183 $$\therefore p' = wx\cos\theta \frac{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}} \quad \text{.....(6.)}$$

Cont. The intensity of the greatest pressure will evidently be $OX = OX'$

$OX : OP :: p_1 : p$ $$\therefore p_1 = p\frac{OX}{OP} = wx\cos\theta \frac{OM(1+\sin\varphi)}{OM(\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)})}$$

$$\therefore p_1 = \frac{wx\cos\theta(1+\sin\varphi)}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}} \quad \text{.....(13.)}$$

The angle which the principal stress makes with the horizon is
$$\psi = 90° - Y'OM = 90° - \tfrac{1}{2}OMQ = 90° - \tfrac{1}{2}(TQM - \theta)$$

$\left(TQM = \text{arc.}\sin\frac{MT}{MQ}\right)$ $$\therefore \psi = \tfrac{1}{2}\left(\theta + 180 - \text{arc.}\sin\frac{\sin\theta}{\sin\varphi}\right) \quad \text{.....(14.)}$$

$$\{\theta = 0\} \qquad p_1 = wx, \quad \psi = 90°$$
$$\{\theta = \varphi\} \qquad p_1 = wx(1+\sin\varphi), \quad \psi = \tfrac{1}{2}(\theta + 90) \quad \text{.....(16.)}$$

The third conjugate stress will be perpendicular to the plane of p, p' & p_1, that is horizontal and parallel to the surface, and its intensity will be the least possible in order to balance p_1, that is to keep p_1 from pushing the earth horizontally. From Eq. (2A)
$$\frac{p''}{p_1} = \frac{OY}{OX} = \frac{1-\sin\varphi}{1+\sin\varphi} \therefore p'' = p_1\frac{1-\sin\varphi}{1+\sin\varphi} = \frac{wx\cos\theta(1-\sin\varphi)}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}} \quad \text{..(10.)}$$

The most important theory of earth pressure after that of Rankine is the one of Prof. Jacob J. Weyrauch who bases his theory on the assumption *"That the forces upon any imaginary plane-section through the mass of earth have the same direction"*.

For a knowledge of this theory, which is an enlargement and extension of that of Coulomb, published 1773, the student is referred to "Retaining-Walls for Earth" by Malverd A. Howe, Wiley N. Y. 1886.

p.330 §186 For equation (2) which in America is usually called the Prismoidal Formula see page 3 of these Notes. It is strictly accurate when the solid is bounded by continuous planes from one end to the other. It is also accurate for a sphere and nearly so for most solids; the greatest inaccuracy occurs when the middle section is very small.

Case III. From eq.(2) $V = \frac{\frac{x}{2}}{3}\left[S_0 + 4S_1 + S_2\right]$

$$S_0 = 2b_0h_0 + sh_0^2, \quad S_1 = 2b_0\frac{h_0+h_2}{2} + s\left[\frac{h_0+h_2}{2}\right]^2, \quad S_2 = 2b_0h_2 + sh_2^2$$

$$V = \frac{x}{6}\left[2b_0h_0 + sh_0^2 + 4\left(2b_0\frac{h_0+h_2}{2} + s\left(\frac{h_0+h_2}{2}\right)^2\right) + 2b_0h_2 + sh_2^2\right]$$

$$= \frac{x}{6}\left[2b_0(h_0+h_2) + s(h_0^2+h_2^2) + 4b_0(h_0+h_2) + s(h_0+h_2)^2\right].$$

$$= x\left[b_0(h_0+h_2) + s.\frac{h_0^2+h_2^2}{6} + s.\frac{h_0^2+2h_0h_2+h_2^2}{6}\right]$$

$$= x\left[b_0(h_0+h_2) + s\,\frac{h_0^2+h_0h_2+h_2^2}{3}\right] \text{..................} (3)$$

$$\text{or } V = x\left[b_0(h_0+h_2) + s\left(\frac{(h_0+h_2)^2}{4} + \frac{(h_0-h_2)^2}{12}\right)\right] \text{...........} (4)$$

When there are a number of sections in succession eq.(1) becomes eq.(5) and eq.(2) becomes eq.(6).

For rapid computation of earthwork as in estimating for an excavation or embankment, especially in preliminary work, see Trautwine's Excavations and Embankments. Wiley N.Y. 1887.

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p.343 §204

Fig. 1

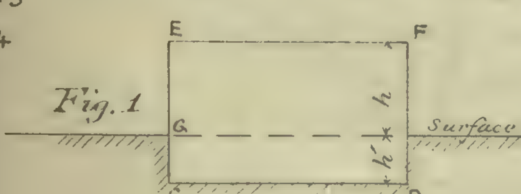
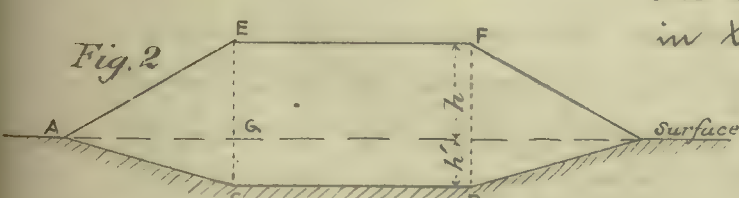


Fig. 2



Consider first the conditions of equilibrium in the earth (Fig. 1) before the excavation and filling. The maximum horizontal thrust in that earth consistent with stability is given by eq.(3A) p.320 §183, viz.

$\frac{p'}{p} = \frac{1+\sin\varphi'}{1-\sin\varphi'}$, where p' is the horizontal and p the vertical pressure.

$\therefore p' = p\,\frac{1+\sin\varphi'}{1-\sin\varphi'}$, is the superior limit of the intensity of

p.343 §204 Cont. the horizontal pressure that may exist in the soft earth at the points C and D

Now the embankment may be made with a limit of heaviness determined by this maximum horizontal thrust, Thus eq. (1A) p.320 § 183 gives

$$\frac{p_1}{p_2} \leq \frac{1+\sin\varphi'}{1-\sin\varphi'}.$$

In this p_2 is horizontal and p_1 vertical and this equation gives the greatest vertical pressure p_1 which is consistent with the horizontal pressure p_2

$\therefore\ p_1 = p_2 \dfrac{1+\sin\varphi'}{1-\sin\varphi'}$. Make $p_2 = p' = p\dfrac{1+\sin\varphi'}{1-\sin\varphi'}$, and we have $p_1 = p\left(\dfrac{1+\sin\varphi'}{1-\sin\varphi'}\right)^2 = \dfrac{p}{k'^2}$, where for brevity $k' = \dfrac{1-\sin\varphi'}{1+\sin\varphi'}$.

For the point D, $p_1 = w(h+h')$ and $p = w'h'$

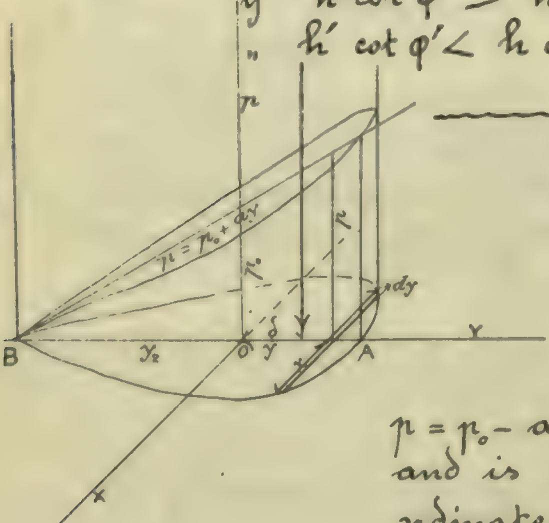
$$\therefore\ w(h+h') = \frac{w'h'}{k'^2} \quad \therefore\ whk'^2 + wh'k'^2 = w'h' \quad \therefore$$

$$h' = \frac{whk'^2}{w'-wk'^2} \text{ ------------------ } (1)$$

The depth h' thus found is on the safe side and if the embankment were as in Fig. 1 it would be stable; giving it the shape of Fig. 2 increases p' (at D) and consequently p_1 might be considerably increased before the softer material would be squeezed out.

The distance AG is taken $= h'\cot\varphi'$ simply for convenience in digging the ditch if it should turn out to be less than $h\cot\varphi$ the latter must be used and we have

if $h'\cot\varphi' > h\cot\varphi$ $AG = h'\cot\varphi'$ .... (2)
" $h'\cot\varphi' < h\cot\varphi$ $AG = h\cot\varphi$



p.378 §236 If the stress be assumed uniformly varying the intensity at any point may be graphically represented as in the accompanying figure as the distance between two planes, $p = p_0 - ay$, where p_0 is the mean pressure and is consequently represented by the ordinate p_0 drawn through the c. of g. of the

p 378 figure of the base.

§236 Since at the point B, $p=0$, we have $0 = p_0 - a y_2$

Cont.

$$\therefore a = \frac{p_0}{y_2}\ ;\ \text{but } p_0 = \frac{P}{A} \quad \therefore a = \frac{P}{Ay} \quad \therefore p = p_0 + \frac{p_0}{y_2}y = \frac{P}{A} + \frac{P}{Ay_2}y$$

$$P\delta = \int_{y_2}^{y_1} y\,p\,x\,dy = \frac{P}{A}\int_{y_2}^{y_1} xy\,dy + \frac{P}{Ay_2}\int_{y_2}^{y_1} xy^2\,dy$$

Since the origin is at the centre of gravity $\int_{y_2}^{y_1} xy\,dy = 0$

$$\therefore \quad P\delta = \frac{P}{Ay_2}\int xy^2\,dy = \frac{PI}{Ay_2}$$

since we have used y_2 to mean the same that the author means by y we have

$$\delta = \frac{I}{Ay} = qh \text{ - } (1.)$$

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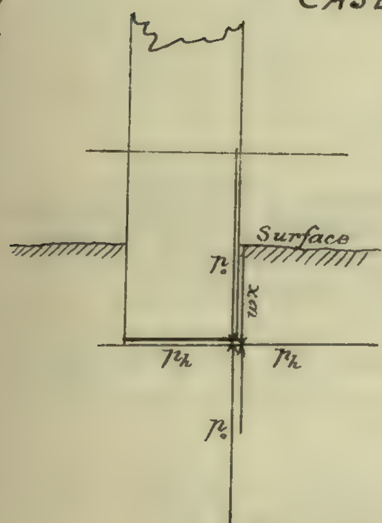
p 379

§237



CASE I Eq. (1A) p 319, §183 $\left\{\frac{p_2}{p_1} \geqq \frac{1-\sin\varphi}{1+\sin\varphi}\right\}$

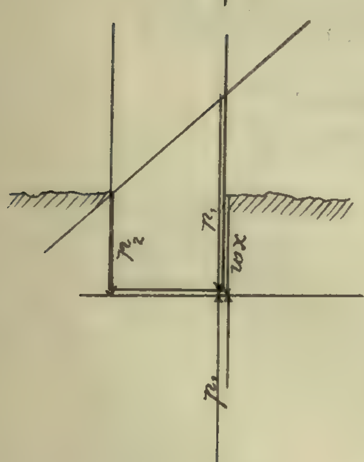
Using the notation of this article we have the least horizontal pressure which will balance p_0 given by the equation

$$\frac{p_h}{p_0} = \frac{1-\sin\varphi}{1+\sin\varphi} \quad \therefore p_h = p_0\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)$$

and the least pressure which will balance p_h is

$$wx = p_h\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right) = p_0\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^2$$

$$\therefore \frac{wx}{p_0} \geqq \left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^2 \text{ - - - - - - - - - - - } (1)$$



CASE II

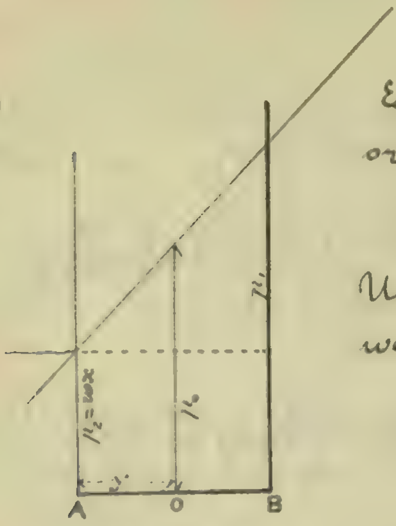
In the same way that we got equation (1) we get Eq. (2)

$$\frac{wx}{p_1} \geqq \left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^2 \text{ - - - - - - - - - - - - - - - } (2.)$$

Equation (3) is only true when the sign of equality is used in eq. (2). The horizontal pressure p_h exists in the earth, and p_2 must be at least sufficient to balance it,

$$\therefore \frac{p_2}{p_1} \geqq \left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^2 \text{ - - - - - - - - - - - - - - - } (4.)$$

p 379 § 237 cont.



SECTION

Eq. 5 may be worked out as was eq.(1) § 236, or otherwise thus.

$$\text{Eq.(7) p. 163 § 106} \left\{ x_0 = \frac{a\int x^2 y\, dx}{P} \right\}$$

Using the notation of the present article we have $\delta = \dfrac{aI}{P}$, but $a = \dfrac{p_0 - p_2}{y}$

$$\therefore \delta = \frac{p_0 - p_2}{y} \cdot \frac{I}{P} \; ; \text{ but } \frac{P}{A} = p_0 \;\therefore\; P = p_0 A$$

$$\therefore \delta = \frac{p_0 - p_2}{p_0} \cdot \frac{I}{Ay} = qh\, \frac{p_0 - p_2}{p_0} \quad \text{.........} \quad (5.)$$

When O is half way between A and B $p_0 = \dfrac{p_1 + p_2}{2}$

$$\frac{wx}{p_0} = \frac{2wx}{p_1 + p_2} = \frac{2wx}{wx\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2 + wx} = \frac{2}{\frac{(1+\sin\varphi)^2 + (1-\sin\varphi)^2}{(1-\sin\varphi)^2}} = \frac{(1-\sin\varphi)^2}{1+\sin^2\varphi}$$

$$= \frac{1 - 2\sin\varphi + \sin^2\varphi}{1 + \sin^2\varphi} = 1 - \frac{2\sin\varphi}{1+\sin^2\varphi}$$

$$\therefore \frac{wx}{p_0} \geq \frac{(1-\sin\varphi)^2}{1+\sin^2\varphi} = 1 - \frac{2\sin\varphi}{1+\sin^2\varphi} \quad \text{.........} \quad (6.)$$

$$\delta = qh\,\frac{p_1 - p_2}{p_1 + p_2} = qh\, \frac{wx\left[\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2 - 1\right]}{wx\left[\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2 + 1\right]} = qh\, \frac{\frac{4\sin\varphi}{(1-\sin\varphi)^2}}{\frac{2(1+\sin^2\varphi)}{(1-\sin\varphi)^2}} = qh\,\frac{2\sin\varphi}{1+\sin^2\varphi}$$

$$\therefore \delta = qh\,\frac{p_1 - p_2}{p_1 + p_2} = qh\,\frac{2\sin\varphi}{1+\sin^2\varphi} \quad \text{.........} \quad (7.)$$

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p.381 § 239

$$\text{Eq.(2) p. 379 § 237} \left\{ \frac{wx}{p_1} \geq \left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^2 \right\}$$

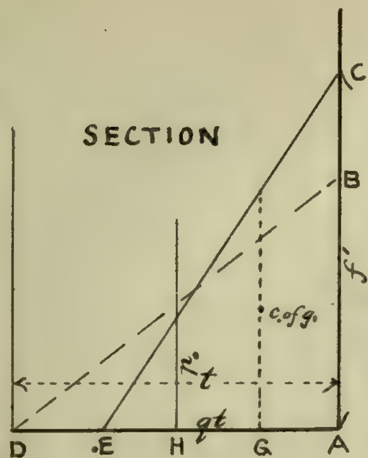
Which in this case becomes

$$\frac{w'x'}{p_1 + wx'} = \left(\frac{1-\sin\varphi'}{1+\sin\varphi'}\right)^2 = k'^2$$

$$\therefore w'x' = k'^2 p_1 + k'^2 wx'$$

$$\therefore x' = \frac{p_1 k'^2}{w' - wk'^2} \quad \text{.........} \quad (1)$$

p. 396
§ 263



Let $AD = t$ = thickness of wall;
H = its central point;
G = required centre of pressure
Lay off $AC = f'$ = limit of pressure.
Draw the triangle ACE so that its area $= \frac{R}{b}$ = pressure per unit of breadth
Now the centre of pressure of ACE is at $G = \frac{1}{3}$ of the distance from A to E

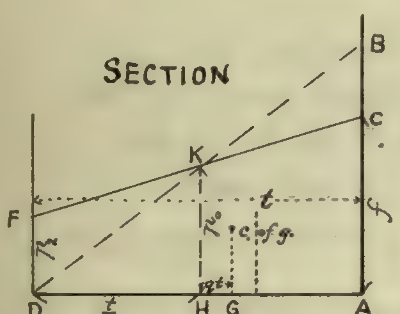
$$AG = AH - HG = \tfrac{1}{2}t - qt$$

$$\therefore AE = 3AG = \tfrac{3}{2}t - 3qt \quad \text{and} \quad ACE = \frac{f'}{2}\left(\tfrac{3}{2}t - 3qt\right) = \frac{R}{b}$$

$$\therefore f' = \frac{2R}{(\frac{3}{2} - 3q)bt} \quad \text{and} \quad q = \frac{1}{2} - \frac{2R}{3f'bt} \quad \text{.........} (1.)$$

In the above case $f' = AC > AB$ (AB is double the mean pressure when the pressure is distributed over the entire base, or equal what would be the maximum pressure if it were zero at D)

But there is another case where $f' = AC < AB$ or $f' < 2p_0$



This case may be solved by using eq. (5) p. 379, § 237, or otherwise thus.

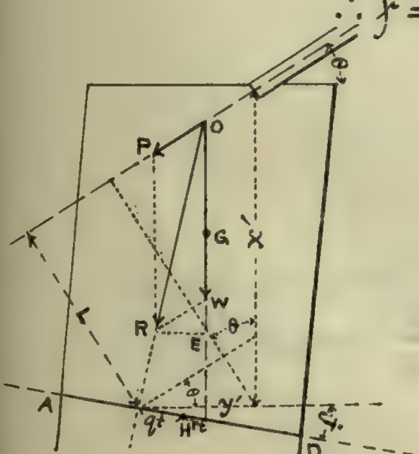
$$\text{Eq. 7 p 163 § 106} \left\{ x_0 = \frac{a\int x^2 y\, dx}{P} \right\}$$

$$\therefore qt = \frac{aI}{R}$$

From the figure we see that $a = \dfrac{f' - p_0}{\frac{1}{2}t}$ and for a rectangle $I = \frac{1}{12}t^3 b$

$$\therefore qt = \frac{f' - p_0}{\frac{1}{2}t} \cdot \frac{t^3 b}{12} \cdot \frac{1}{R} = \left(f' - \frac{R}{bt}\right)\frac{t^2 b}{6R} = \frac{1}{6}\frac{bf't^2}{R} - \frac{t}{6}$$

$$\therefore f' = (1 + 6q)\frac{R}{bt} \quad \text{and} \quad q = \frac{1}{6}\left(\frac{bf't}{R} - 1\right) \quad \text{..........} (1\text{A}.)$$



$$M = W(q \pm r)t\cos j \quad \text{..........} (2.)$$

$$W = n\, w h b t \quad \text{..........} (3)$$

$$M = n(q \pm r)\cos j\; w \cdot h b t^2 \quad \text{..........} (4.)$$

$$L = x'\cos\theta - y'\sin\theta$$

$$PL = P(x'\cos\theta - y'\sin\theta) \leq M \quad \text{..........} (5.)$$

p.399 §263 Cont.

$$\tan\ ROW = \frac{RE}{OE} = \frac{RE}{OW+WE} = \frac{P\cos\theta}{P\sin\theta+W}$$

$$\therefore ROW = \text{arc}\tan\frac{P\cos\theta}{P\sin\theta+W}$$

$$\therefore \text{arc.}\tan\frac{P\cos\theta}{P\sin\theta+W} - j \leqq \varphi \quad \ldots\ldots\ldots\ldots \quad (6.)$$

---

p.401 §264 From eq.(2) we obtain eq.(4) viz.

$$n w h_0 b_0 t_0 = P\left(\frac{\cos\theta}{\tan\varphi} - \sin\theta\right) \quad \ldots\ldots\ldots \quad (4.)$$

when $n = 0$ &c. $h_0 bt = \frac{P}{w}\left(\frac{\cos\theta}{\tan\varphi} - \sin\theta\right) = \frac{P}{w}\,\frac{\cos(\varphi+\theta)}{\sin\varphi}$

from eq. 6 $qwbt^2 = P\cos\theta \therefore wbt = \frac{P\cos\theta}{qt} \therefore h_0 = qt\cdot\frac{\cos(\varphi+\theta)}{\sin\varphi\cos\theta}$

substituting for t its value from eq. 6

$$h_0 = qt\,\frac{\cos(\varphi+\theta)}{\sin\varphi\cos\theta} = \frac{\cos(\varphi+\theta)}{\sin\varphi}\sqrt{\frac{qP}{wb\cos\theta}} \quad \ldots\ldots\ldots \quad (7.)$$

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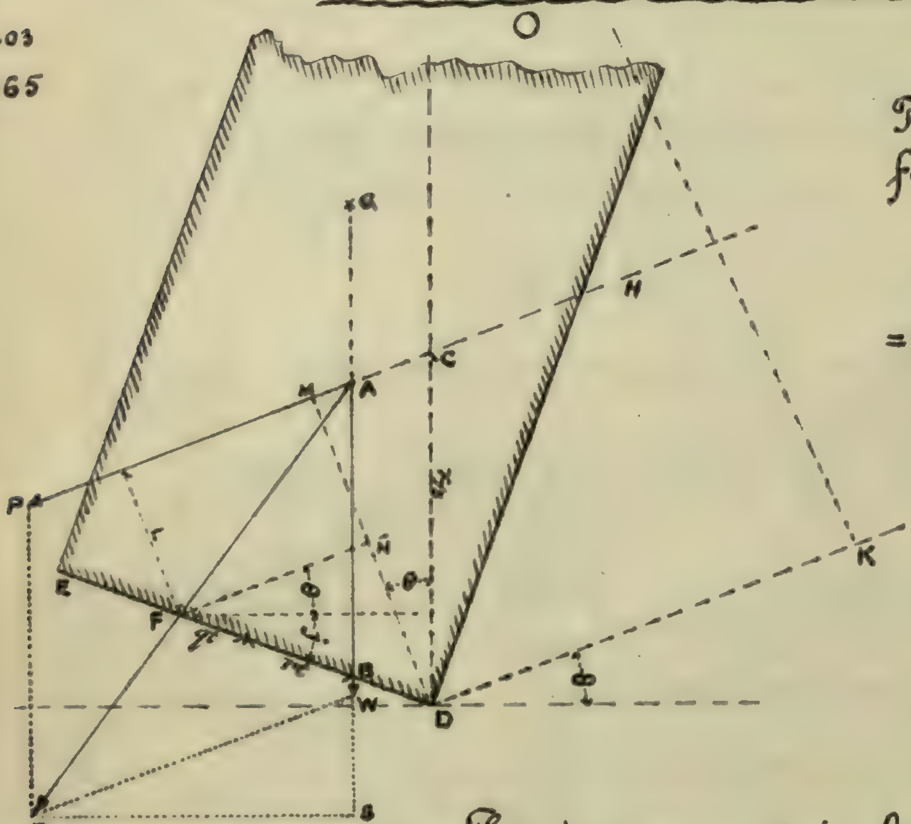
p.403 §265



The lever arm of the force P is $L = MN$

$= MD - ND$

$= CD\cos\theta - FD\sin(\theta+j)$

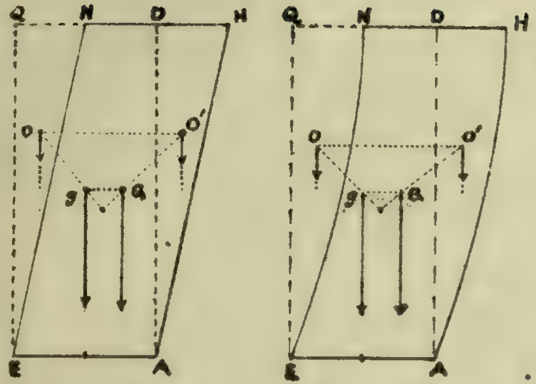
$= \frac{x}{3}\cos\theta - (q+\frac{1}{2})t\sin(\theta+j)$

p.405 §266 The trigonometrical reduction in eq. 2 is as follows

$$\frac{1-\sin\varphi}{1+\sin\varphi} = \frac{1-\cos(90-\varphi)}{1+\cos(90-\varphi)} = \frac{\sin^2\frac{1}{2}(90-\varphi)}{\cos^2\frac{1}{2}(90-\varphi)} = \tan^2\left(\frac{90-\varphi}{2}\right)$$

$$\therefore \frac{t}{x} = \tan\frac{90-\varphi}{2}\sqrt{\frac{w'}{6qw}} \quad \ldots\ldots\ldots \quad 2$$

p.406 §268



By referring to Case VIII §104 we see that the distance

$$Gg = OO' \frac{EQN}{EADQ} \text{ where } OO' =$$

the distance of C. of G. of EQN in its first position from the C. of G. in its last position = (in this case) t.

$$\therefore Gg = t\,\frac{EQN}{tx} = \frac{EQN}{x} \quad \text{.................} (1)$$

---

p.408 §269 Volume of masonry in the counterforted wall

$$= h\,(bt + cT)$$

Volume of masonry in the "equivalent uniform wall"

$$= h(b+c)t_1 = \sqrt{(b+c)(bt^2 + cT^2)} \cdot h$$

Squaring and rejecting common terms, we have

$$2Tt \gtreqless T^2 + t^2$$

Subtracting $2Tt$ from both members, we get

$$0 \gtreqless (T - t)^2$$

Since $T-t$ can never equal zero the second member of the expression is always greater than the first; or the amount of masonry in the counterforted wall is less than that in the "equivalent uniform wall".

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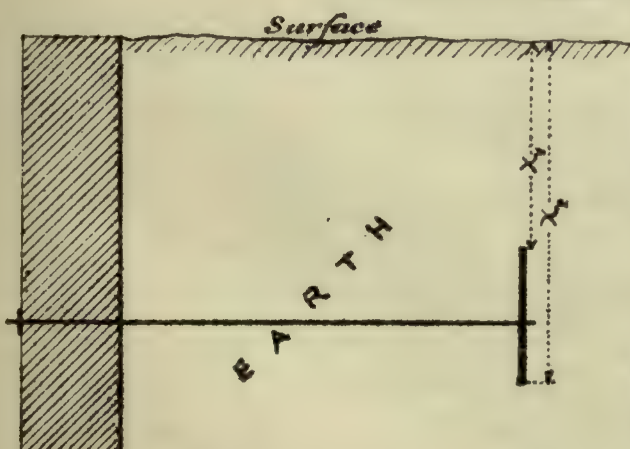
p.410 §272



If the plates are imbedded vertically, the pressure of the earth behind them against them will be

$$wx\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)$$

Now they can exert a pulling force on the earth in front of them equal the maximum value of the ratio $\frac{p'}{wx}$ or =

p 410 $w'x\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)$ Of this max. pull, however, a part =
§ 272
Cont. $w'x\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)$ is equal the thrust received by the plate from the earth behind and is therefore not available for holding up the retaining wall.

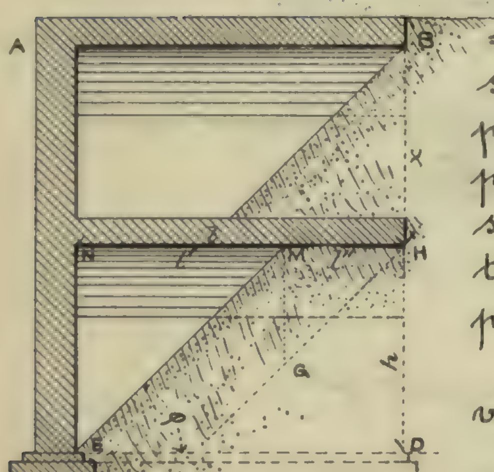
Therefore the net value of h, the horizontal pull is

$$w'x\left[\frac{1+\sin\varphi}{1-\sin\varphi}-\frac{1-\sin\varphi}{1+\sin\varphi}\right]$$

$$= w'x\,\frac{4\sin\varphi}{1-\sin^2\varphi} = wx\,\frac{4\sin\varphi}{\cos^2\varphi}$$

$$H = w'\int_{x_1}^{x_2} x\,\frac{4\sin\varphi}{\cos^2\varphi}dx = w'\,\frac{4\sin\varphi}{\cos^2\varphi}\int_{x_1}^{x_2} x\,dx = w'\,\frac{x_2^2-x_1^2}{2}\cdot\frac{4\sin\varphi}{\cos^2\varphi} \quad\ldots(1)$$

p 412
§ 274



The pressure on the particle at G $= w\cos\varphi\cdot MG$ and, as pressure is the same in every part of a layer parallel to a conjugate plane; the particle at H bears upward with the same force as the one at G [note that when $\theta=\varphi$ in earth, the conj. pressures = each other.
$\therefore\ MG\cdot w\cos\varphi$ = conj. as well as vertical pressure at G]

Resolve this pressure and the horizontal component $= MG\cdot w\cos^2\varphi$

This is the force that balances the horizontal thrust of the earth at H; which last

$$= wx\cdot\frac{1-\sin\varphi}{1+\sin\varphi}$$

Calling $l = NM + MH = l' + l''$, we have

$$MG = l''\tan\varphi \quad\therefore\ l''\cdot\tan\varphi\cdot w\cos^2\varphi = wx\,\frac{1-\sin\varphi}{1+\sin\varphi}$$

$$\text{or } l'' = \frac{x\cot\varphi}{\cos^2\varphi}\cdot\frac{1-\sin\varphi}{1+\sin\varphi} = \cot\varphi\,\frac{x}{(1+\sin\varphi)^2}$$

and $l' = h\cot\varphi$. $\quad\therefore\ l'+l'' = l = \cot\varphi\left[h+\frac{x}{(1+\sin\varphi)^2}\right]$ .... (1)

$$h = l\tan\varphi - \frac{x}{(1+\sin\varphi)^2} \quad\ldots\ldots\ldots\ldots (2)$$

p 412 $\left\{ \frac{t}{x} = \sqrt{\left(\frac{w'}{6qw} \cdot \frac{1-\sin\varphi}{1+\sin\varphi}\right)} \right\}$ Eq. 2, p 405, § 266.
§ 275

In determining this equation the pressure against a units length of the wall was taken as $\left\{P = \frac{w_1 x^2}{2}\right\}$ Eq. 1, p 403.

The pressure to be used in the buttressed wall is $P' = P\frac{B}{b} = \frac{B}{b}\frac{w_1 x^2}{2} = \frac{B}{b} \cdot \frac{x^2}{2} \cdot w' \frac{1-\sin\varphi}{1+\sin\varphi}$.

$$\therefore \frac{T}{x} = \sqrt{\frac{B}{b}\frac{w'}{6qw} \cdot \frac{1-\sin\varphi}{1+\sin\varphi}}$$

$$\therefore \frac{T}{t} = \sqrt{\frac{B}{b}} \quad \text{or} \quad T = t\sqrt{\frac{B}{b}} \quad \therefore b = B\frac{t^2}{T^2} \text{ - - - - - - - - - - } (1)$$

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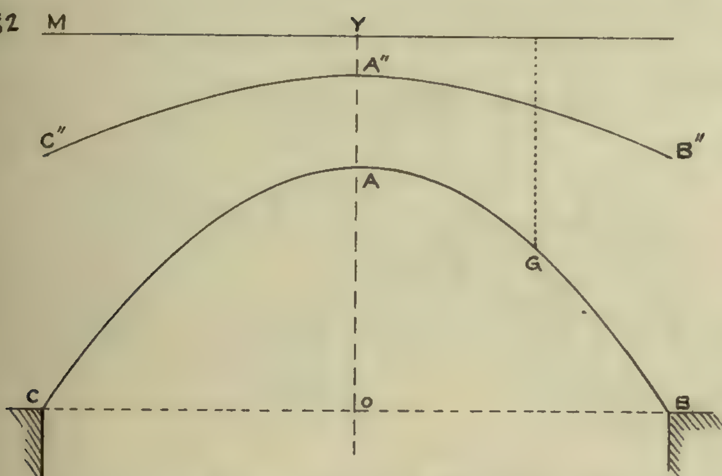
p 418
§ 282



For a discussion of the Transformed catenary as applied to arches see pp. 50, 51 of these Notes, to which may be added the following.

The extrados of the Transformed Catenary need not be the directrix MY. It may be another transformed catenary having the same directrix.

<u>To illustrate</u>. Suppose the weight of a unit of the material between CAB and MY = w. Then the intensity of vertical pressure at any point, as G of CAB, = wy. If a heavier building material were used, this vertical pressure could be brought upon G by a less height of it. Let a unit, weight of the heavier material, = w', and let $w' = \frac{3}{2}w$; then a column of the heavier material whose height = $\frac{2}{3}y$ would give the same pressure as the entire column of the lighter; or

$$wy = \tfrac{2}{3}w'y$$

At each point of CAB measure upward $\frac{2}{3}$ of the vertical ordinate and through these points draw the curve C"A"B".

The upper surface of the load may have this form and CAB still be balanced under the forces.

Since the equation of CAB is

$$y = \frac{y_0}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)$$ That of C"A"B" is evidently

$$y' = \frac{\frac{1}{3} y_0}{2}\left(\varepsilon^{\frac{x}{m}} + \varepsilon^{-\frac{x}{m}}\right)$$

p 418
§ 282 The principle of this example is general.

When the extrados is the *Transformed Catenary* we must note that, since in all the formulæ of § 131 w = weight corresponding to a unit of surface of the space between CAB and MY, we must make, in those formulæ.

$$w = nw'$$

where w' is the weight of a unit of vol. of building material and $n = \frac{AA''}{AY}$. If there be voids w' becomes kw_1, where w_1 is the weight of a unit vol. of the building material and k has the meaning assigned to it in the text, that is the ratio of the solid building to the whole volume.

Then $\quad w = nkw_1$ . . . . (2)

---

p 419
§ 283 $\left\{\begin{array}{l} \rho_0 = a + \dfrac{a^2}{2x_0} \\ \rho_1 = a - \dfrac{a^2}{2(x_0+a)} \end{array}\right\}$ Eq 12 p 211 and $\left\{x_0 = a\dfrac{a^3}{b^3 - a^3}\right\}$ Eq. 11 p 211

Substituting for x_0 its value $\rho_0 = a + \dfrac{a^2}{2\dfrac{a^4}{b^3-a^3}} = a + \dfrac{b^3 - a^3}{2a^2}$

$$= \frac{2a^3 + b^3 - a^3}{2a^2} = \frac{a^3 + b^3}{2a^2} = \frac{a}{2} + \frac{b^3}{2a^2} = \frac{a}{2}\left(1 + \frac{b^3}{a^3}\right)$$

Similarly we obtain $\quad \rho_1 = \frac{a}{2}\left(1 + \frac{a^3}{b^3}\right)$

---

p 420
§ 284 $\left\{x_0 = a\dfrac{a^3}{b^3 - a^3}\right\}$ $\therefore \quad b^3 - a^3 = \dfrac{a^4}{x_0} \quad \therefore \quad b^3 = a^3\left(\dfrac{a}{x_0} + 1\right)$

$$\therefore b = a\left(\frac{x_0 + a}{x_0}\right)^{1/3}$$

$\left\{b = y_1 + \dfrac{y_1^2}{30a}\right\}$ Eq. 11 p. 211 Since b and y_1 are nearly the same, and since this is only an approximate formula any way, we may write it $b = y_1 + \dfrac{b^2}{30a} \therefore y_1 = b - \dfrac{b^2}{30a}$.

Then $c = \dfrac{s}{y_1} = \sqrt{\dfrac{p_x}{p_y}}$ see Eq. 1 p. 213 § 137 where p_y is the intensity of the horizontal pressure and p_x of the vertical pressure.

It will be remembered that in earth pressure

p420 §284 Cont. the horizontal intensity p_y, supposing the surface level is $\leqq p_x \frac{1+\sin\varphi}{1-\sin\varphi}$ and $\geqq p_x \frac{1-\sin\varphi}{1+\sin\varphi}$

p_y being the passive pressure will be just what is necessary to resist the horizontal component of the thrust of the arch provided it is within the above limits. For this reason we have considerable latitude in choosing s in Equation 1

p421 §285 By referring to Figs. 1 & 2 p.56 of these Notes, it will be seen that tension from without is required from the "point of rupture" D to the crown A.

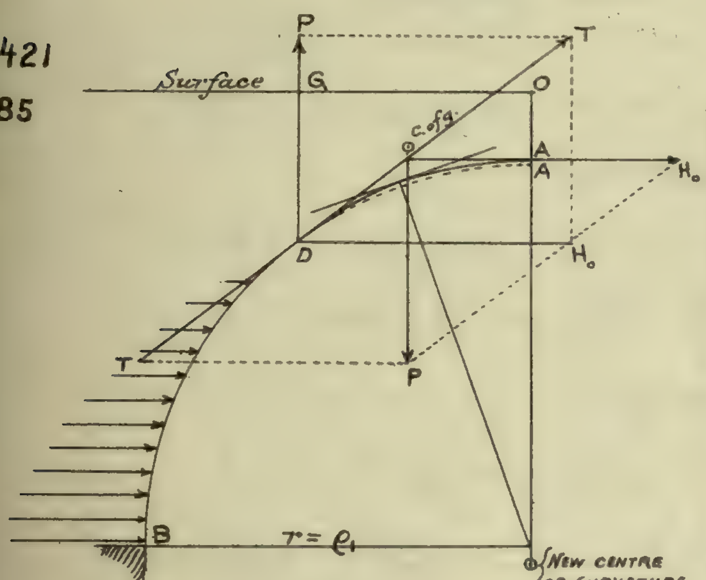


Fig. 1

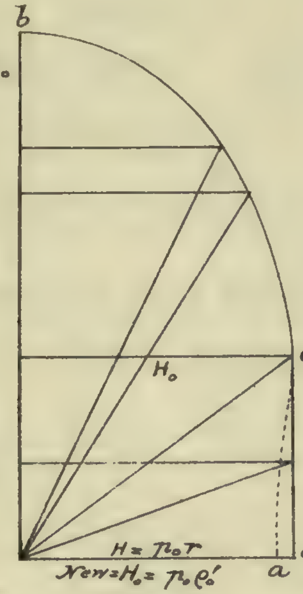


Fig. 2

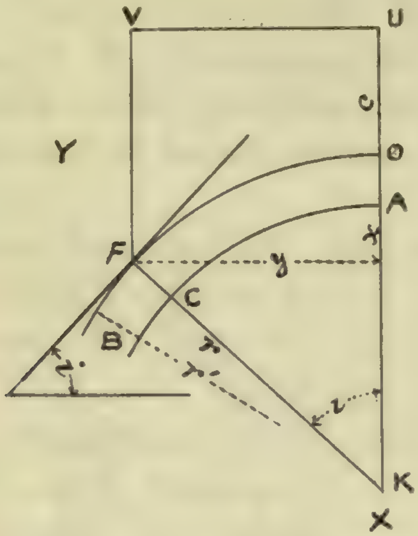
Since, ordinarily, backing cannot pull, the linear rib must be changed above D and given such a shape that it will have a constant horizontal component of thrust = H_0

This changed linear rib is shown here in *Fig. 1*, and the corresponding diagram of forces in *Fig. 2*, the original being shown in each case by the broken lines.

The new radius at the crown is $\rho_0' = \frac{H_0}{p_0}$. The new crown is found by drawing a tangent to the curve thus. In *Fig. 1* from the point where a vertical through the c.of g. of DGOA intersects DT draw a horizontal line, this intersects the line OA at A', the new crown. Other points may be found by completing the diagram of forces, by drawing a line from d parallel to ob, then working backwards from the diagram of forces we can get the shape of the new linear rib. The whole linear rib may now be projected to near

p421 the centre of the arch stones and the arch will be
§285 stable if it falls within the middle third
Cont.

p 423
§ 286



$$\left\{ P = wr^2\left[(1+m)\sin i - \frac{\sin i.\cos i}{2} - \frac{i}{2}\right]\right\}$$

Eq (a) p. 58 Notes

in this equation to make it apply here $mr' = c \therefore m = \frac{c}{r'}$; $w = w'$; $r = r'$

For load above extrados

$$P = w'r'^2\left[(1+\tfrac{c}{r'})\sin i - \frac{\sin i.\cos i}{2} - \frac{i}{2}\right]$$

Weight of arch ring ACFO $= w\,\frac{r'^2 - r^2}{2}i$

$\therefore$ For total load

$$P = w'r'^2\left[(1+\tfrac{c}{r'})\sin i - \frac{\sin i.\cos i}{2} - \frac{i}{2}\right] + w\frac{r'^2-r^2}{2}i$$

The horizontal component of the thrust $H = P\cot i = P\frac{\cos i}{\sin i}$

$$\therefore H = \frac{\cos i}{\sin i}\left[w'r'^2\left\{(1+\tfrac{c}{r'})\sin i - \frac{\sin i.\cos i}{2} - \frac{i}{2}\right\} + w\frac{r'^2-r^2}{2}.i\right] \therefore$$

$$H = w'r'^2\left\{(1+\tfrac{c}{r'})\cos i - \frac{\cos^2 i}{2} - \frac{i.\cot i}{2}\right\} + w\frac{r'^2-r^2}{2}.i.\cot i \quad \therefore$$

$$H_0 = w'r'^2\left\{(1+\tfrac{c}{r'})\cos i_0 - \frac{\cos^2 i_0}{2} - \frac{i_0.\cot i_0}{2}\right\} + w\frac{r'^2-r^2}{2}.i_0.\cot i_0 \quad \text{.....}(2)$$

$$\left\{ \S 138, \text{p. } 214 \ (4) \quad p_y = -\frac{dH}{dx}\right\} ; \quad dx = r'.\sin i.di \quad \therefore p_y = -\frac{1}{r'\sin i}.\frac{dH}{di}$$

$$\therefore p_y = -\frac{w'r'^2}{r'\sin i}.\frac{d}{di}\left\{(1+\tfrac{c}{r'})\cos i - \frac{\cos^2 i}{2} - \frac{i\cot i}{2}\right\} - \frac{w(r'^2-r^2)}{r'\sin i}.\frac{d}{di}\left\{\frac{i\cot i}{2}\right\}$$

but
$$d\left(\frac{\cos^2 i}{2}\right) = \tfrac{1}{2}.2\cos i(-\sin i)di = -\sin i.\cos i.di$$
$$d\left(\frac{i.\cot i}{2}\right) = \tfrac{1}{2}\left(\cot i.di - \frac{i\,di}{\sin^2 i}\right) = \tfrac{1}{2}\left(\frac{\cos i}{\sin i} - \frac{i}{\sin^2 i}\right)di = \tfrac{1}{2}\left(\frac{\sin i.\cos i - i}{\sin^2 i}\right)di$$

$$\therefore p_y = -\frac{w'r'^2}{r'\sin i}\left\{(1+\tfrac{c}{r'})(-\sin i) - (-\sin i.\cos i) - \left(-\frac{i - \sin i.\cos i}{2\sin^2 i}\right)\right\}$$
$$- \frac{w(r'^2-r^2)}{r'\sin i}\left(-\frac{i - \sin i\cos i}{2\sin^2 i}\right) \therefore$$

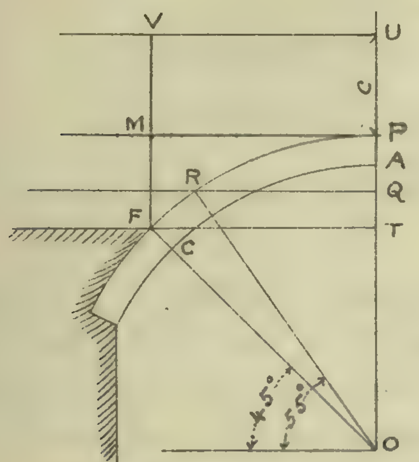
$$p_y = \frac{w'r'^2}{r'}\left\{(1+\tfrac{c}{r'}) - \cos i - \frac{i - \sin i.\cos i}{2\sin^3 i}\right\} + \frac{w(r'^2-r^2)}{r'}.\frac{i - \sin i.\cos i}{2\sin^3 i}$$

when $p_y = 0$, multiplying by r' we have

$$0 = w'r'^2\left\{(1+\tfrac{c}{r'}) - \cos i_0 - \frac{i_0 - \sin i_0\cos i_0}{2\sin^3 i_0}\right\} + w(r'^2-r^2).\frac{i_0 - \sin i_0\cos i_0}{2\sin^3 i_0}$$

p 423 $$O = \frac{w'}{w}\left\{1 - \cos i_0 - \frac{i_0 - \sin i_0 \cos i_0}{2\sin^3 i_0}\right\} r'^2 + \frac{w'}{w} c r' + \frac{i_0 - \sin i_0 \cos i_0}{2\sin^3 i_0} r'^2 - \frac{i_0 - \sin i_0 \cos i_0}{2\sin^3 i_0} r^2$$

§ 286 Cont. $$O = \left\{\frac{w'}{w}(1 - \cos i_0) + \left(1 - \frac{w'}{w}\right)\frac{i_0 - \sin i_0 \cos i_0}{2\sin^3 i_0}\right\} r'^2 + \frac{w'}{w} c r' - \frac{i_0 - \sin i_0 \cos i_0}{2\sin^3 i_0} r^2 \quad \ldots (1)$$



The spandril FVUP is composed of two parts
1st The rectangle MVUP
2nd " surface FMP

$FT = OT = r'\sqrt{\tfrac{1}{2}}$ $\quad PT = PO - TO = r'(1 - \sqrt{\tfrac{1}{2}})$

Hence area of sector OFP
$= \frac{1}{8} \cdot 2\pi r' \cdot \frac{r'}{2} = .3927\, r'^2$

Area of triangle $OFT = \frac{(r'\sqrt{\frac{1}{2}})^2}{2} = 0.25\, r'^2$

$\therefore$ " Segment $FTP = 0.1427\, r'^2$

Area of rectangle $FMPT = \overline{PT} \times \overline{FT} = 0.2071\, r'^2$

$\therefore$ Area $FMP = 0.0644\, r'^2$

Again $MP = FT = r'\sqrt{\tfrac{1}{2}}$ $\therefore$ area $MVUP = cr'\sqrt{\tfrac{1}{2}} = 0.7071\, cr'$

Also area of ring $FCPA = \frac{1}{8}(\pi r'^2 - \pi r^2) = 0.3927\,(r'^2 - r^2)$

$\therefore$ For the whole weight of ACFVU which is nearly equal to H_0 we have

$$H_0 \text{ nearly} = w'r'(0.0644\, r' + 0.7071\, c) + .03927\, w(r'^2 - r^2) \quad \ldots (4)$$

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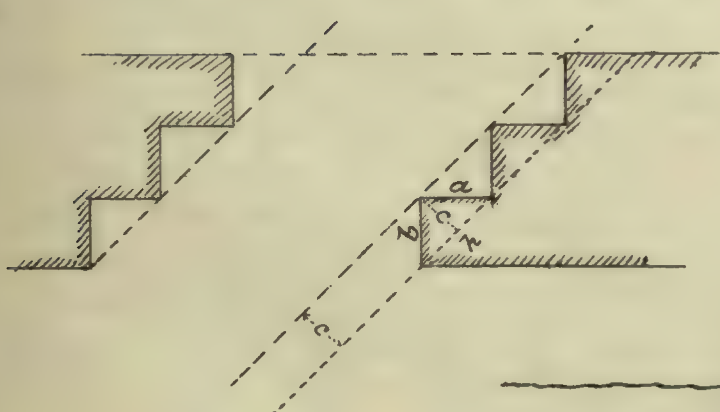
p 432
§ 296



The amount of contraction is the distance marked c. By inspection of the figure
$h = \sqrt{a^2 + b^2}$
$a : c :: h : b$

$$\therefore c = \frac{ab}{h} = \frac{ab}{\sqrt{a^2+b^2}} \quad \ldots\ldots\ldots (1)$$

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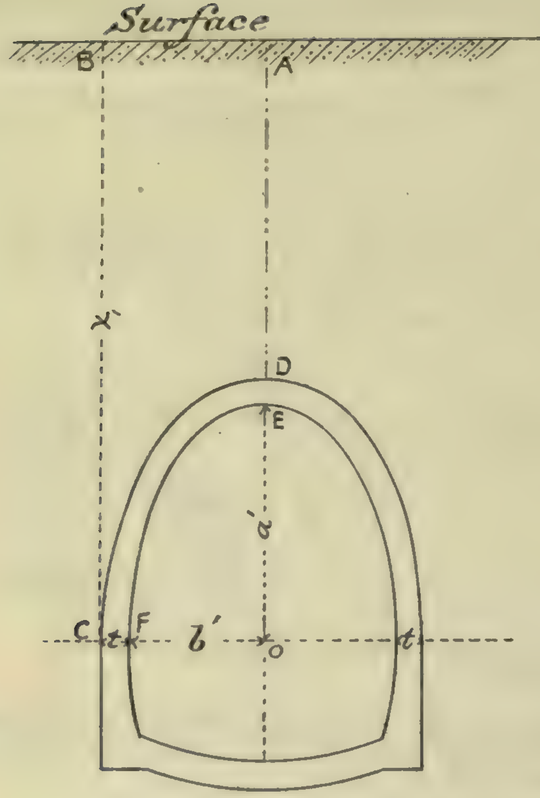
p 435
§ 297



Length of tunnel considered = 1 ft.

The load which must be supported at CF = weight AEFCB

$= ABCO - EFO$

$= wx_1(b'+t) - \frac{1}{4}\pi a'b'w$

$= wx_1(b'+t) - w(0.8\,a'b')$

$$\therefore\; q = \frac{w\{x_1(b'+t) - 0.8\,a'b'\}}{t}; \;\dots\dots\; (2)$$

The difference between the weight of the half arch and that of an equal volume of earth is small and not taken into account.

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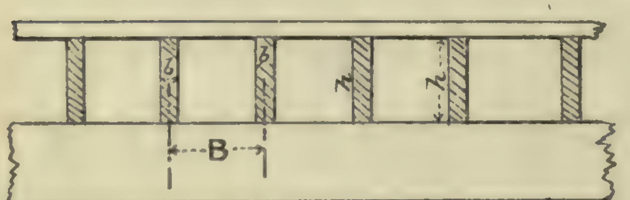
p 467
§ 336



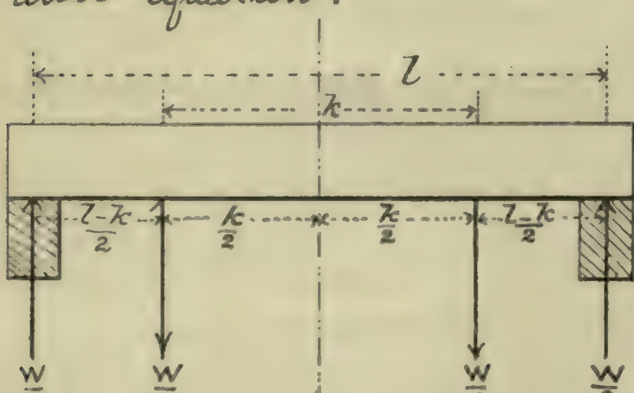
$\{M_0 = mWl = nfbh^2\}$ p. 253 § 162 Eq. 6

$n = \frac{1}{6}$; $m = \frac{1}{8}$; $f = 1000$ lbs pr. square inch $\therefore$

$$\frac{1000\,bh^2}{6} = \frac{150\,Bl}{8\times144}\cdot l = \frac{150\,Bl^2}{8\times144} \;\text{-------------}\; (1)$$

This equation is gotten by supposing all dimensions to be in inches, but it is evident that it will still be true if we change the dimensions to feet on each side.

For stone roadways substitute the weight of a cu. ft. of the stone roadway 250 lbs pr. sq. ft. for the 150 in the above equation.



$\{nfbh^2 = mWl\}$ $\therefore$

$$\frac{1000\,bh^2}{6} = \frac{1}{2}W\cdot\frac{l-k}{2} = \frac{W(l-k)}{4} \;\dots (3)$$

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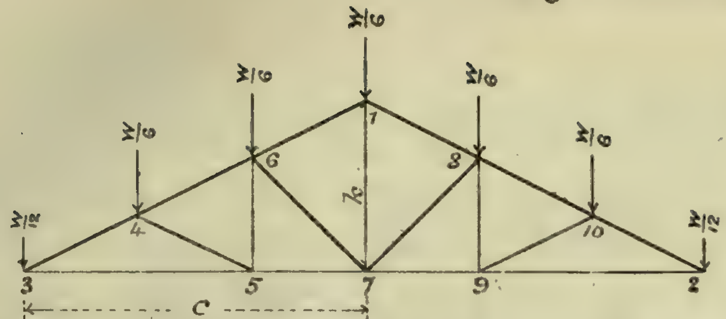

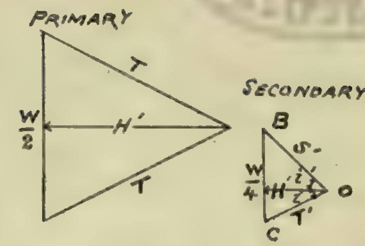

The load will be distributed as shown in the figure $\frac{1}{6}W$ resting at each joint of the rafters and $\frac{1}{12}W$ resting on each wall

Tertiary Truss 3 4 5 The points 3 and 5 act as points of support for the $\frac{W}{6}$ which rests at 4 and since the points 3 and 5 are equally distant from 4 horizontally $\frac{1}{12}W$ will rest at each so that the rod 5 6 must sustain this $1/12\,W$ and add it to the $\frac{1}{6}W$ resting directly at 6 making the total load on the truss 367 $\frac{1}{12}W + \frac{1}{6}W = \frac{3}{12}W = \frac{1}{4}W$

Secondary Truss 367 Total load $\frac{1}{4}W$, Since the point of support 3 is twice as far away, horizontally, from 6 as the other point of support 7 is $\therefore\ \frac{1}{3}\cdot\frac{1}{4}W$ rests at 3 and $\frac{2}{3}\cdot\frac{1}{4}W$ must be held up by the rod 17 due to the left side of the truss, an equal amount comes from the right side so that the total amount held up by the rod 17 is $2\cdot\frac{2}{3}\cdot\frac{1}{4}W = \frac{1}{3}W$ add this to the load resting directly at 1 we have $\frac{1}{3}W + \frac{1}{6}W = \frac{1}{2}W$ as the load on the primary truss

It will be observed that if we suppose the load uniformly distributed the load on each truss is $\frac{1}{2}$ of the total load directly above that truss

From similar triangles $H : \frac{W}{4} :: c : k \quad \therefore\ H = \frac{W}{4}\cdot\frac{c}{k}$ & $T = \sqrt{H^2 + \frac{W^2}{16}}$
From the diagram of forces of the secondary truss

$$\left.\begin{aligned} CH &= OH \tan i \\ BH &= OH \tan i' \end{aligned}\right\} \therefore\ CB = OH(\tan i + \tan i') \quad \therefore\ OH = \frac{CB}{\tan i + \tan i'}$$

$$\therefore\ H' = \frac{\frac{W}{4}}{\dfrac{2/3\,k}{2/3\,c} + \dfrac{2/3\,k}{1/3\,c}} = \frac{\frac{W}{4}}{\frac{k}{c} + \frac{2k}{c}} = \frac{Wc}{12k}$$

$$\left.\begin{aligned} P &= H'\tan i = \frac{Wc}{12k}\cdot\frac{k}{c} = \frac{W}{12} \\ Q &= H'\tan i' = \frac{Wc}{12k}\cdot\frac{2k}{c} = \frac{W}{6} \end{aligned}\right\} \therefore\ H' = \frac{W}{12k}\ ;\ T' = \sqrt{H'^2 + \frac{W^2}{144}}\ ;\ S' = \sqrt{H'^2 + \frac{W^2}{36}} \quad\cdots(2)$$

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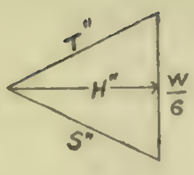
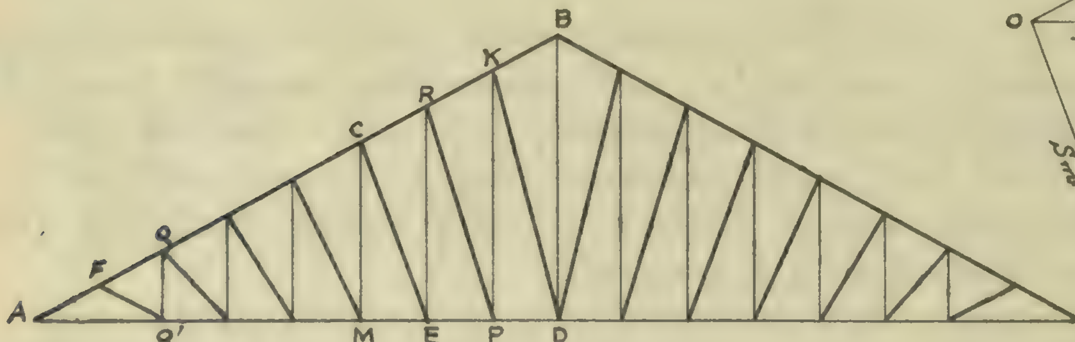
Cont.



$$H'' : \frac{W}{12} :: \frac{c}{3} : \frac{k}{3} \quad \therefore \quad H'' = \frac{Wc}{36} \div \frac{k}{3} = \frac{Wc}{12k} \quad \therefore$$

$$H'' = \frac{Wc}{12k}\ ; \quad T'' = S'' = \sqrt{H''^2 + \frac{W^2}{144}} \quad \text{- - - - - - - - - - - - -} \quad (3)$$

$$H + H' = \frac{Wc}{4k} + \frac{Wc}{12k} = \frac{4Wc}{12k} = \frac{Wc}{3k}\ ; \quad H + H' + H'' = \frac{Wc}{3k} + \frac{Wc}{12k} = \frac{5}{12}\frac{Wc}{k} \quad (4)$$



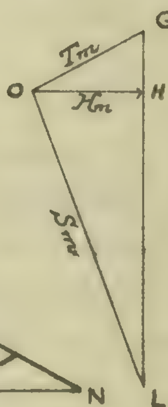


Diagram of Forces for Truss ACE

General Case

The amount of load supported at each joint is $\frac{W}{2n}$; the amount resting directly at A and N is $\frac{W}{4n}$

To find the total load to be supported by any truss we begin at either end as A. The load resting at F will be supported by the wall at A and the suspension rod QQ'. Since the points of support are equally distant, horizontally, the pull on the rod QQ' will be $\frac{1}{2}\frac{W}{2n}$

1^{st} truss	Load $= \frac{W}{2n}$	Pull on 1^{st} rod	$= \frac{1}{2}\frac{W}{2n}$
2^{nd} ,,	,, $= \frac{W}{2n} + \frac{1}{2}\cdot\frac{W}{2n} = \frac{3}{2}\cdot\frac{W}{2n}$	,, ,, 2^{nd} ,,	$= \frac{2}{3}\cdot\frac{3}{2}\cdot\frac{W}{2n} = \frac{2}{2}\cdot\frac{W}{2n}$
3^{rd} ,,	,, $= \frac{W}{2n} + \frac{2}{2}\cdot\frac{W}{2n} = \frac{4}{2}\cdot\frac{W}{2n}$	,, ,, 3^{rd} ,,	$= \frac{3}{4}\cdot\frac{4}{2}\cdot\frac{W}{2n} = \frac{3}{2}\cdot\frac{W}{2n}$
4^{th} ,,	,, $= \frac{W}{2n} + \frac{3}{2}\cdot\frac{W}{2n} = \frac{5}{2}\cdot\frac{W}{2n}$	,, ,, 4^{th} ,,	$= \frac{4}{5}\cdot\frac{5}{2}\cdot\frac{W}{2n} = \frac{4}{2}\cdot\frac{W}{2n}$
μ^{th} ,,	,, $= \frac{\mu+1}{2}\cdot\frac{W}{2n}$	,, ,, μ^{th} ,,	$= \frac{\mu}{2}\cdot\frac{W}{2n}$
$\mu+1^{th}$,,	,, $= \frac{W}{2n} + \frac{\mu}{2}\cdot\frac{W}{2n} = \frac{\mu+2}{2}\cdot\frac{W}{2n}$	,, ,, $(\mu+1)^{th}$,,	$= \frac{\mu+1}{\mu+2}\cdot\frac{\mu+2}{2}\cdot\frac{W}{2n} = \frac{\mu+1}{2}\cdot\frac{W}{2n}$

Hence we see the law and can write the load for any truss as the truss ACE. This load turns out to be $\frac{1}{2}$ of the load between A and C on the one side and $\frac{1}{2}$ of that between C and R on the other.

The number of sections in AC is $(n-m)$ $\therefore$ the weight at

p 472 $C = \frac{n-m+1}{2} \cdot \frac{W}{2n}$; similarly we get the weight $K = (n-1)^{th}$

§ 339 truss to be $\frac{(n-1)+1}{2} \cdot \frac{W}{2n} = \frac{W}{4}$. the portion of this supported

Cont. by the central rod BD is $2\,\frac{n-1}{2} \cdot \frac{W}{2n} = (n-1)\frac{W}{2n}$. Since an equal amount comes from the other side of the truss we have the total load resting on the rod BD $= 2 \cdot \frac{n-1}{2} \cdot \frac{W}{2n}$; to get the total load resting at B we must add to this, $\frac{W}{2n}$, the load resting directly at B $\therefore\ 2 \cdot \frac{n-1}{2} W + \frac{W}{2n} = (n-1+1)\frac{W}{2n} = \frac{W}{2}$ is the load on the primary truss

If C is the m^{th} joint from the centre then the load as given above is $\frac{n-m+1}{2} \cdot \frac{W}{2n}$ OGL is the triangle of forces for this truss. In this the small triangle OGH is similar to ACM and so is OHL to CME

$$
\begin{array}{ll}
\text{Hence} & CM : GH :: AM : OH \\
\text{and} & CM : ME :: HL : OH \\
\text{whence} & GH \cdot AM = ME \cdot HL
\end{array}
$$

and $GH : HL :: ME : AM :: CR : AC :: 1 : n-m$

$\therefore\quad GH : GH + HL :: 1 : n-m+1$

$\therefore\quad GH = \frac{1}{n-m+1} \cdot GL = \frac{1}{n-m+1} \cdot \frac{n-m+1}{4n} \cdot W = \frac{W}{4n}$

[Since GL = weight at C $= \frac{n-m+1}{4n} \cdot W$]

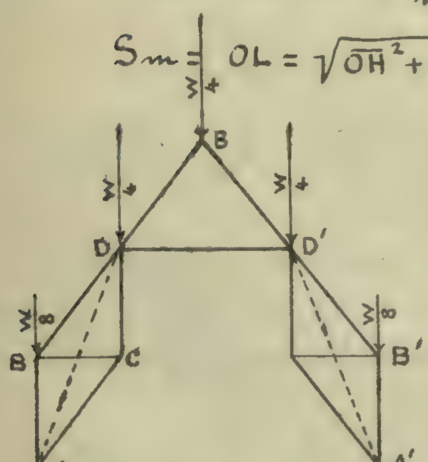
$OH = H_m = \frac{W}{4n} \cdot \frac{c}{k}$ which is independent of m and $\therefore$ constant,

$$T_m = OG = \sqrt{\overline{GH}^2 + \overline{OH}^2} = \sqrt{\left(\frac{W}{-n}\right)^2 + H_m^2} = \sqrt{\left(\frac{W}{4n}\right)^2 + \left(\frac{W}{4n}\right)^2 \frac{c^2}{k^2}} = \frac{W}{2n}\sqrt{1 + \frac{c^2}{k^2}} \quad (6$$

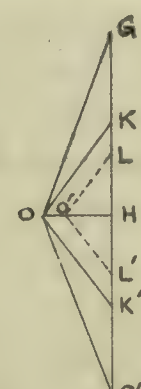
$$HL = GL - GH = \frac{n-m}{n-m+1} \cdot GL = \frac{n-m}{n*m+1} \cdot \frac{n-m+1}{4n} W = \frac{n-m}{4n} W$$

$$S_m = OL = \sqrt{\overline{OH}^2 + \overline{HL}^2} = \sqrt{H_m^2 + \frac{W^2}{16n^2} \cdot (n-m)^2} = \frac{W}{4n}\sqrt{\frac{c^2}{k^2} + (n-m)^2} \cdots\cdots (7)$$

IV



Gothic Roof-Trusses



The drawing in the text Fig. 212 has several pieces which should be left out or in the discussion the load should be considered distributed at twice as many points.

The only pieces which Rankine considers as acting are shown in the accompaning drawing.

The diagram of forces for the main truss ADD′A′ is constructed by drawing $GG' = \frac{3}{4}W$ the total load, from G and G′ draw

p474 lines parallel to DA and D'A' meeting in O. Then will OH
§339 represent the horizontal thrust in the piece DD' and OG
Cont. the oblique thrust in the direcection of the dotted line
DA. Draw OK || BD || AC; then K will be the middle
point of HG. The triangle of forces OGK gives the forces
acting at D on the lower pieces

$$GK = \tfrac{3}{16}W = \text{thrust on } DC$$

$$OK = \quad \text{thrust on } BD$$

The triangle KOH represents the forces at C and also at B
before the $\frac{W}{8}$ resting directly at B is added

$$OH = \text{tension in } BC$$

$$OK = \text{thrust on } AC$$

$$KH + \frac{W}{8} = \text{thrust on } BA$$

O'LL' is the diagram of forces for the truss DED' where
$LL' = \frac{W}{4}$ = total load on this truss

$$\therefore HL = \tfrac{2}{3}HK \quad \therefore \quad O'H = \tfrac{2}{3}OH \text{ and } O'L = \tfrac{2}{3}OK$$

O'H = tension in DD' which is therefore equal numerically
to 2/3 of thrust or compression which comes from its being
a member of the primary truss.

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$$M = HL$$

$$\left\{ \S\,162 \text{ — } M = nfbh^2 = \tfrac{1}{6}fbh^2 \right\} \therefore$$

$$f = \frac{6M}{bh^2} \quad \therefore \quad f' = \frac{6M}{bh^2} + \frac{H}{bh} \quad \text{and } b = \frac{6M}{f'h^2} + \frac{H}{f'h} = \left(\frac{6M}{h^2} + \frac{H}{h}\right) \div f' \text{ --- } (1 \,\&\, 2)$$

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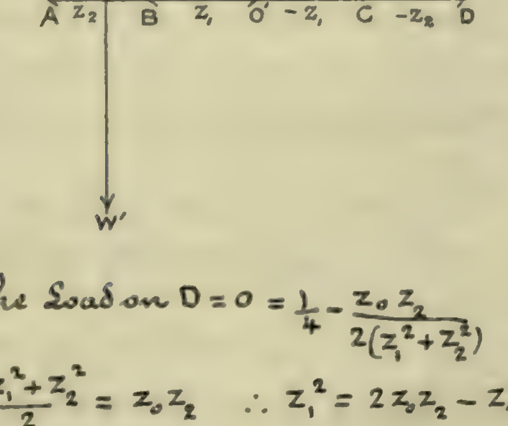
I Load Unequal Transversely



$$W'z_0 = 2(az_1 \cdot z_1 + az_2 \cdot z_2) \quad \therefore$$

$$W'z_0 = 2a(z_1^2 + z_2^2) \quad \therefore \quad a = \frac{W'z_0}{2(z_1^2 + z_2^2)}$$

$$\left.\begin{array}{lll}
\text{Load on A — } & & W'\left[\frac{1}{4} + \frac{z_0 z_2}{2(z_1^2 + z_2)^2}\right] \\
\text{,, ,,} & B & W'\left[\frac{1}{4} + \frac{z_0 z_1}{2(z_1^2 + z_2^2)}\right] \\
\text{,, ,,} & C & W'\left[\frac{1}{4} - \frac{z_0 z_1}{2(z_1^2 + z_2^2)}\right] \\
\text{,, ,,} & D & W'\left[\frac{1}{4} - \frac{z_0 z_2}{2(z_1^2 + z_2^2)}\right]
\end{array}\right\} (1)$$

Make Load on D = 0 = $\frac{1}{4} - \frac{z_0 z_2}{2(z_1^2 + z_2^2)}$

$$\therefore \frac{z_1^2 + z_2^2}{2} = z_0 z_2 \quad \therefore z_1^2 = 2z_0 z_2 - z_2^2 \text{ ---------------------------- } (2)$$

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II Load Unequal Longitudinally

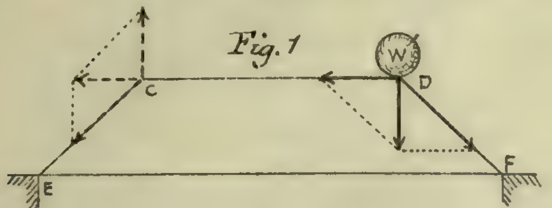

Fig. 1

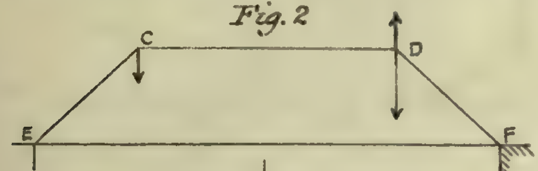

Fig. 2

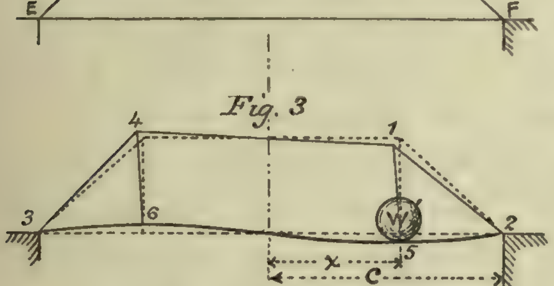

Fig. 3

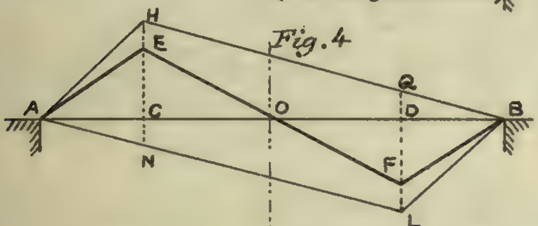

Fig. 4

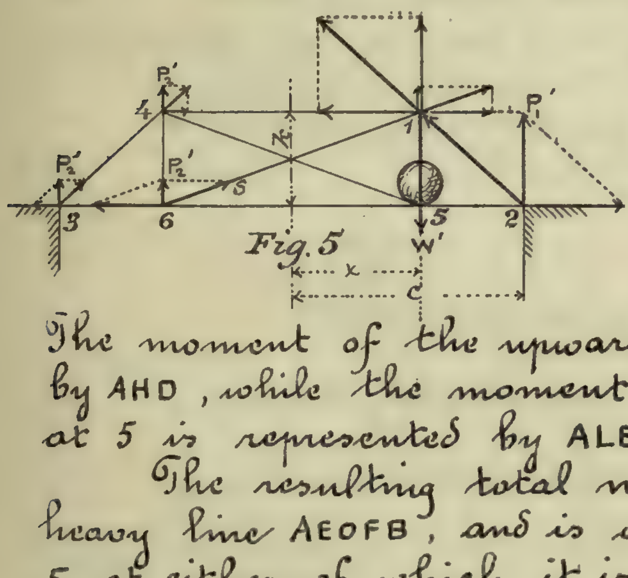

Fig. 5

Let ECDF Fig. 1 be the truss. If a weight rests at D, the frame can be kept in position either by an upward force = W' acting at D or by a downward force = [illegible] acting at C or, as shown in Fig. 2, by by an upward force acting at D $= \frac{W'}{2}$ and a downward force acting at C $= \frac{W'}{2}$

In Fig. 3 the unequal load is sustained by a stiff beam 3, 6, 5, 2. The beam is shown bent and the truss slightly distorted in consequence. The beam bends slightly as the load comes on the point 1 which causes that point to come down; simultaneously the point 4 goes up, causing an upward pull on the beam at 6. Since the bending is in proportion to the load and since 4 must go up as fast as 1 comes down, the upward force applied to the beam at 6 must be the same as the downward force applied at 5.

Fig. 4 shows the diagram of moments.

The moment of the upward force $(=\frac{W'}{2})$ acting at 6 is represented by AHD, while the moment of the downward force $(=\frac{W'}{2})$ acting at 5 is represented by ALB.

The resulting total moment is represented by the heavy line AEOFB, and is a maximum at the points 6 and 5 at either of which it is

$$\frac{W'}{2}\,\frac{c+x}{2c}(c-x) - \frac{W'}{2}\,\frac{c-x}{2c}(c-x) = \frac{W'x(c-x)}{2c} \quad \text{--------------} \quad (3)$$

Where diagonal bracing is used Fig. 5 shows the action of all the forces, caused by W', and the stress on each piece

$$W' : P_1' : P_2' :: 2c : c+x : c-x \quad \therefore \quad P_2' = \frac{W'(c-x)}{2c}$$

The thrust in the diagonal 6,1, $= \quad S = P_2' \sec 461 = P_2' \frac{6,1}{4,6}$

$$= \frac{W'}{2c}(c-x) \times \frac{\sqrt{4x^2+k^2}}{k} = W'\,\frac{c-x}{2ck}\sqrt{4x^2+k^2} \quad \text{----------} \quad (4)$$

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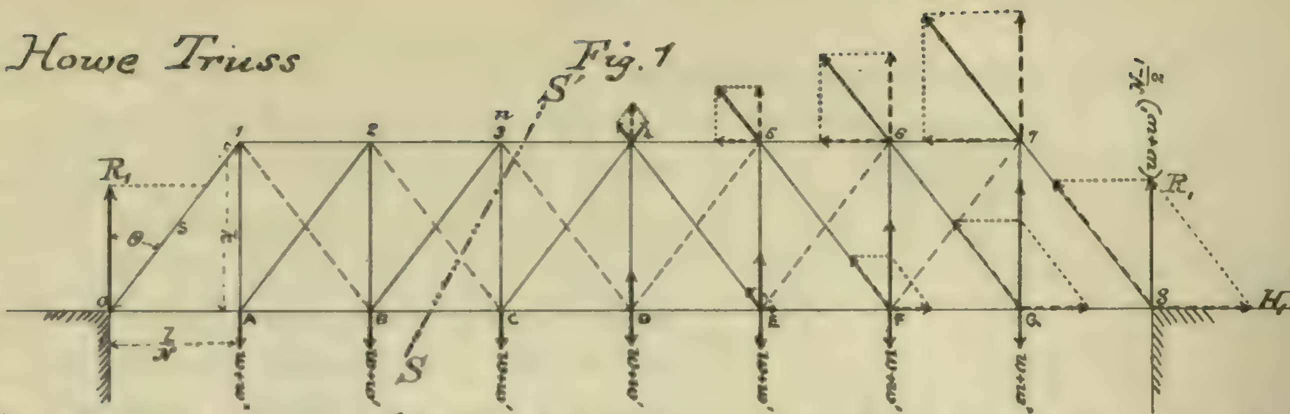
Fig. 1



The above figure shows the action of the forces on one half of the bridge when the entire bridge is covered by the travelling load. The bridge is symmetrical and consequently the forces will be the same on the other half so they are not drawn.

The parallelograms of forces may also be taken to represent the action of the permanent load if we use the proper scale.

At every point on the bridge the bending moment is greatest when the travelling load covers the entire bridge so that we get the maximum stresses in the top and bottom chords. By inspecting Fig. 1 it will be seen that the compression in 12 = the tension in OA.

Starting at the left abutment with $H_1 = R_1 \tan\theta = (w+w')\frac{N-1}{2}\cdot\frac{l}{Nk}$ we see, by inspecting the above parallelograms of forces, that at each point where the load rests H receives an increment which increment decreases each time by $(w+w')\frac{l}{Nk}$

We may now write out the horizontal stresses

$$H_1 = (w+w')\frac{N-1}{2}\cdot\frac{l}{Nk} = \frac{w+w'}{k}\cdot l\cdot\frac{N-1}{2N}$$

$$H_2 = \frac{w+w'}{k}\cdot l\cdot\frac{N-1}{2N} + w+w'\cdot\frac{l}{Nk}\left(\frac{N-1}{2}-1\right) = \frac{w+w'}{k}\cdot l\cdot\frac{2(N-2)}{2N}$$

$$H_3 = (w+w')\frac{l}{Nk}\left(\frac{2(N-2)}{2}+\left(\frac{N-1}{2}-2\right)\right) = \frac{w+w'}{k}\cdot l\cdot\frac{3(N-3)}{2N}$$

$$\therefore H_n = \frac{w+w'}{k}\cdot l\cdot\frac{n(N-n)}{2N} \quad \ldots\ldots \quad (1)$$

$$H_{n+1} = (w+w')\frac{l}{Nk}\left(\frac{n(N-n)}{2}+\frac{N-1}{2}-n\right) = \frac{w+w'}{k}\cdot l\cdot\frac{(n+1)[N-(n+1)]}{2N}$$

Since $\frac{n(N-n)+N-1-2n}{2} = \frac{nN-n^2+N-1-2n}{2} = \frac{(n+1)N-(n+1)^2}{2} = \frac{(n+1)[N-(n+1)]}{2}$

Beyond the middle the law does not hold since the diagonals slope towards the left abutment instead of away from it.

For the vertical rods we have

$1^{st} = V_1' = R_1 = (w+w')\frac{N-1}{2}$

$2^{nd} = V_2' = (w+w')\frac{N-1}{2} - (w+w') = (w+w')\left(\frac{N-1}{2}-1\right)$

p478 $3^{rd} = V_3' = (w+w')\left(\frac{N-1}{2} - 1\right) - w + w' \quad = (w+w')\left(\frac{N-1}{2} - 2\right)$

§343 $n^{th} = V_n' \quad = (w+w')\left[\frac{N-1}{2} - (n-1)\right] = w+w'\left(\frac{N+1}{2} - n\right)$ } A

Cont. In none except the first is the value the maximum that can ever occur in the piece

From the above values we obtain the stresses in the diagonals by multiplying by $\sec\theta$

$$\left.\begin{aligned} &1^{st}\text{ diagonal} = T_1 = R_1 \sec\theta = (w+w')\frac{N-1}{2}\cdot\frac{\sqrt{k^2+\frac{l^2}{N^2}}}{k} = (w+w')\frac{N-1}{2k}\sqrt{k^2+\frac{l^2}{N^2}} \\ &2^{nd} = T_2' = V_2'\sec\theta = (w+w')\left(\frac{N+1}{2} - 2\right)\sqrt{k^2+\frac{l^2}{N^2}} \\ &n^{th}\quad T_n' = V_n'\sec\theta = (w+w')\left(\frac{N+1}{2} - n\right)\sqrt{k^2+\frac{l^2}{N^2}} \end{aligned}\right\} B$$

Again only in the 1^{st} diagonal does this give the max. that can occur

We can get the same formulæ by taking a section, S – S′, Fig. 1, cutting the n^{th} vertical and the n^{th} division of both the upper and the lower boom

We have

$$\Sigma X = 0 = R_1 - (n-1)(w+w') + V_n' \quad \text{--------------} (a)$$

$$\Sigma Y = 0 = H_n - H_n \quad \text{----------------------} (b)$$

$$\Sigma M = 0 = R_1\, n\frac{l}{N} - (n-1)(w+w')\frac{n}{2}\cdot\frac{l}{N} + H_n k \quad \text{-----} (c)$$

The moments in the above equation being taken about the point numbered n (3 in Fig. 1)

From (a)

$$0 = (w+w')\frac{N-1}{2} - (n-1)(w+w') + V_n'$$

$$\therefore V_n' = (w+w')\left(\frac{N+1}{2} - n\right)$$

From (c)

$$H_n k = (w+w')\frac{l}{N}\left(\frac{N-1}{2}\cdot n - \frac{n^2-n}{2}\right) = (w+w')\frac{l}{N}\cdot\frac{n(N-n)}{2}$$

$$\therefore \quad H_n = \frac{w+w'}{k}\cdot l\cdot\frac{n(N-n)}{2N} \quad \text{-----------------} (1)$$

The max. shearing force occurs when the longer section is loaded. Hence the max. stress in the n^{th} vertical occurs, if $n < \frac{N}{2}$, when the load is on n and covers the right hand end of the bridge – In this case the counterbraces cut by the plane $S-S'$ have no stresses in them

The same loading gives T_n the max stress in the diagonal leaning towards the centre whose upper end is at n

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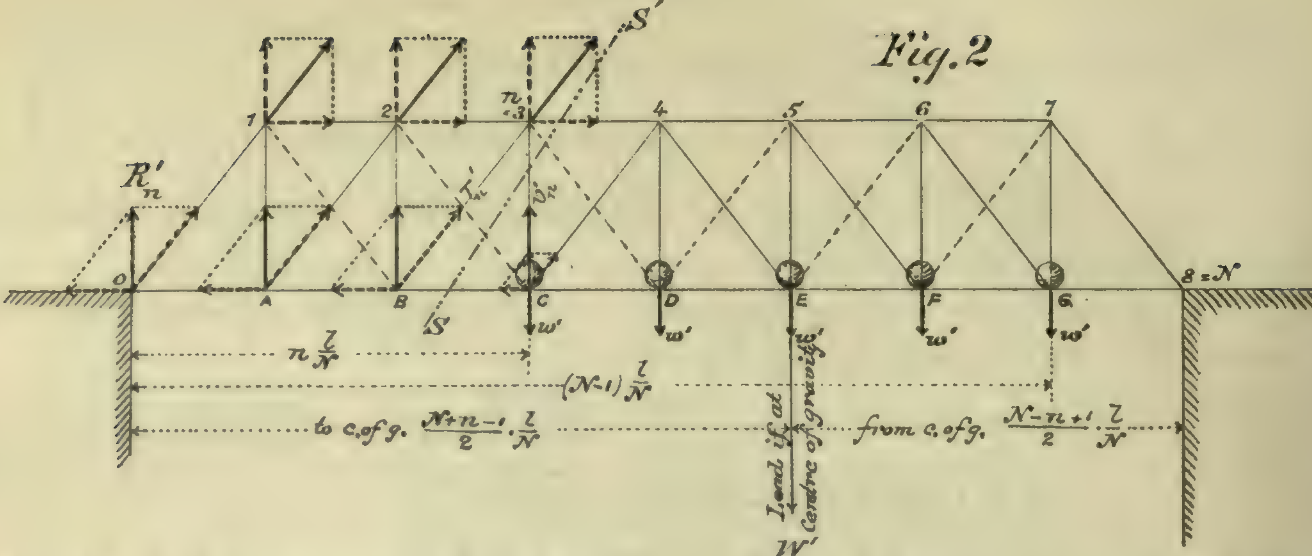
Cont.



In the above figure the bridge is shown loaded at n and beyond. This loading produces forces shown by the parallelograms, the active force at any joint being shown as a full line while resulting components are shown as broken lines.

The forces produced by the travelling load are the only ones shown in Fig. 2; those produced by the permanent load if represented would be like those shown in Fig. 1.

The stress in the n^{th} vertical is made up of two parts one from the permanent load obtained by writing w for $(w+w')$ in equations (A) $v_n = w\left(\frac{N+1}{2} - n\right)$

The other is equal to the resistance of the left abutment coming from the travelling load $R'_n : W' :: \frac{N-n+1}{2}\cdot\frac{l}{N} : l$

$$\therefore R'_n = v'_n = \left[N-1-(n-1)\right] w' \times \frac{N-n+1}{2N} = w'\frac{(N-n)(N-n+1)}{2N} \quad \ldots\ d$$

Up to the middle rod $\therefore$

$$V_n = v_n + v'_n = w\left(\frac{N+1}{2} - n\right) + w'\frac{(N-1)(N-n+1)}{2N} \quad \ldots\ldots\ (2)$$

It will be seen from Fig. 1 that for the middle

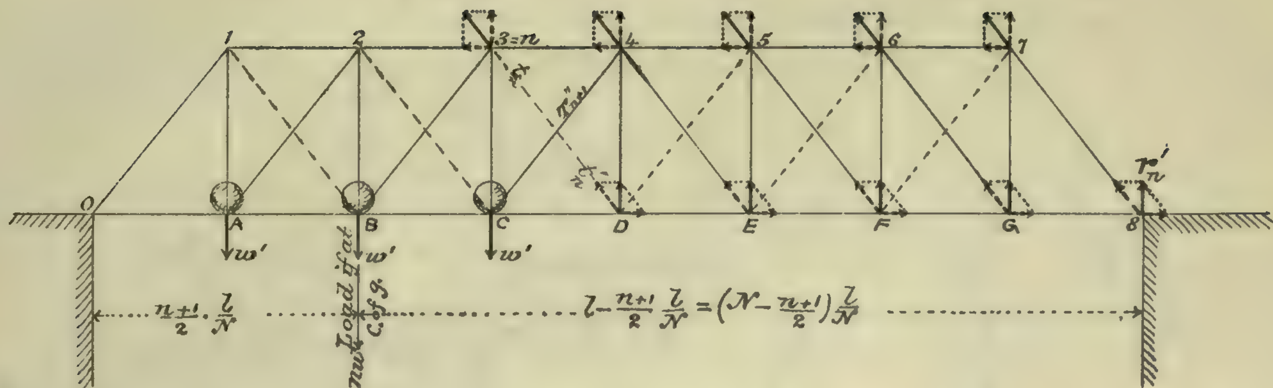
$$v_m = w \text{ and from equation } (d)$$

$$v'_m = \frac{w'\left(N-\frac{N}{2}\right)\left(N-\frac{N}{2}+1\right)}{2N} = w'\,\frac{1}{4}\cdot\left(\frac{N}{2}+1\right)$$

$$\therefore V_{\frac{N}{2}} = w + \frac{w'}{4}\cdot\left(\frac{N}{2}+1\right) \quad \ldots\ldots\ (3)$$

$$T_n = V_n\,\frac{s}{k} = w\frac{s}{k}\left(\frac{N+1}{2} - n\right) + w'\frac{s}{k}\,\frac{(N-1)(N-n+1)}{2N} \quad \ldots\ldots\ (4)$$

p 480
§ 343
Cont.



The maximum stress in diagonals leaning away from the centre as t_n in 30 comes when the shorter end of the bridge is loaded but since the shearing force in this case differs in direction from that produced by the permanent load the stress in the diagonal arising from the travelling load must be diminished by the stress that is produced by the permanent load in the other diagonal in the same panel $t_n = t'_n - T''_{n+1}$

From equations (B) writing w for $w+w'$

$$T_{n+1} = w\left[\frac{N+1}{2} - (n+1)\right]\frac{s}{k} = \frac{ws}{k}\left(\frac{N-1}{2} - n\right)$$

but $\quad T'_n : nw' :: \frac{n+1}{2}\,\frac{l}{N} : l \quad \therefore\; T'_n = \frac{n(n-1)w'}{2N} \;\therefore\; t'_n = \frac{n(n-1)w'}{2N}\cdot\frac{s}{k}$

$$\therefore\; t_n = -T''_{n+1} + t'_n \;=\; -\frac{ws}{k}\left(\frac{N-1}{2} - n\right) + \frac{w's}{k}\cdot\frac{n(n-1)}{2N} \;\dots\dots\dots\; (5)$$

In the foregoing discussion the load has been supposed to rest on the lower chord. If it rested above the stresses would be the same as those already found excepting that in the verticals. The max. stress in the n^{th} vertical would then occur when the load was on $n+1, n+2 \times\times\times$ & $N-1$ but not on n. The c. of g. of the travelling load would then be at a distance from the Right abutment $= \frac{N-n}{2}\cdot\frac{l}{N}$

the resistance of the Left abutment which comes from the travelling load is $\quad v'_n : (N-n-1)w' :: \frac{N-n}{2}\cdot\frac{l}{N} : l \quad \therefore\; v'_n = \frac{w'(N-n-1)(N-2)}{2N}$

For the permanent load $v_n = w\frac{N-1}{2} - nw = w\left(\frac{N+1}{2} - n\right)$

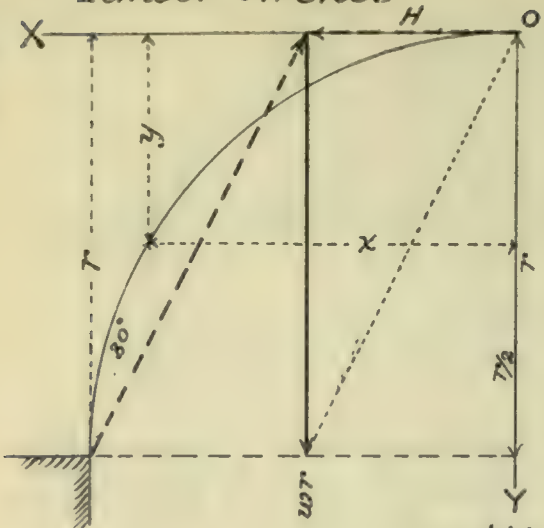
Load on top $\quad \therefore\; V_n = v_n + v'_n = w\left(\frac{N-1}{2} - n\right) + w'(N-n)\frac{N-n-1}{2N} \qquad 2'$

Load on bottom $\quad V_n = w\left(\frac{N+1}{2} - n\right) + w'(N-n)\,\frac{N-n+1}{2N}$

The difference of which is $w + w'\frac{N-n}{N}$ instead of $w+w'$ as stated in the text.

Timber Arches

p.481
§345



$$Hr = wr\cdot\frac{r}{2} \quad \therefore$$

$$H = \frac{wr}{2} \quad \text{.......................} (1)$$

$$x^2 = (2r - y)\,y$$

$$M = wx\cdot\frac{x}{2} - Hy \quad \therefore$$

$$M = \frac{w}{2}(2ry - y^2) - \frac{wr}{2}\,y$$

To find the value which will make M a max.

$$dM = 2r\,dy - 2y\,dy - r\,dy \quad \therefore$$

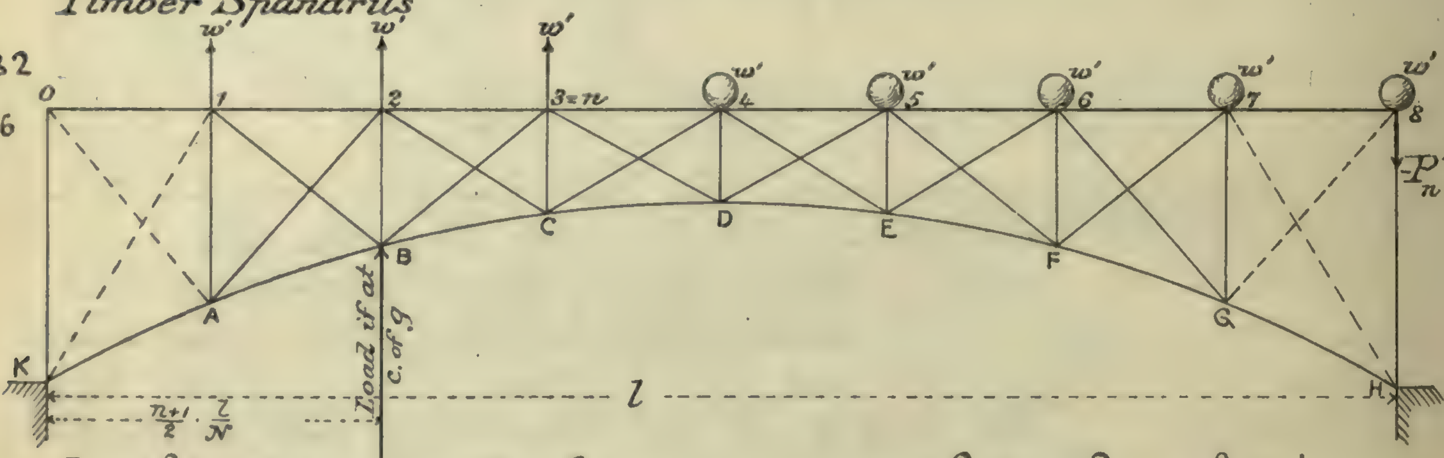
$\frac{dM}{dy} = 0 = r - 2y$ ∴ when $y = \frac{r}{2}$ the moment in the rib is a maximum. This value of y occurs at 30° above the springing ∴

$$M_o = \frac{w}{2}\left[\left(2r\cdot\frac{r}{2} - \frac{r^2}{4}\right) - r\frac{r}{2}\right] = \frac{w}{2}\left(\frac{2r^2}{2} - \frac{r^2}{4} - \frac{r^2}{2}\right) \quad \therefore$$

$$M_o = \frac{1}{8}wr^2 \quad \text{.......................} (2)$$

Timber Spandrils

p.482
§346



In this discussion Rankine assumes that the rib is parabolic and that if either half is uniformly loaded, either with the permanent load only, or with both the permanent and the travelling, it will be in the same condition that it would be if the other half were similarly loaded; and will give the same horizontal thrust at the crown.

If then more than ½ of the bridge is loaded, as shown in the Figure, on the unloaded side the uprights must push down just as much as they would do if it was also loaded. In other words the arched rib on the

p.482 unloaded side gives an upward push $= w'$ at each unloaded
§346 joint thus producing a shearing force which must be resisted
Cont. by the diagonals.

This upward thrust would $= nw'$ and the center of the parallel forces would be at a distance $\frac{n+1}{2}\cdot\frac{l}{N}$ from the left abutment. The resistance of the right abutment will give the shearing stress which must be resisted by the diagonal bracing beyond n. This would throw the stress in 4C

$$-P'_n : -nw' :: \frac{n+1}{2}\cdot\frac{l}{N} : l$$

$$\therefore \quad P'_n = -w'\,\frac{n(n+1)}{2N}$$

And the stress in the diagonal 4C $= T_n = nw'\frac{s}{k}\cdot\frac{n+1}{2N}$ ....(1)
where $s = 4C$ and $k = 3C$

If the bridge were loaded up to and including n and unloaded beyond P'_n would have the same numerical value but different direction, that is, the resistance would be an upward push. The stress would now fall in the diagonal 3D $= T_n = nw'\,\frac{s}{k}\,\frac{n+1}{2N}$ where $s = 3D$ and $k = 4D$

If the n^{th} vertical ever has to act as a strut it will be when the load comes to and includes the $n-1$ joint but is not on the n^{th}. The shearing stress will then be

$$P'_{n-1} : (n-1)w' :: \frac{n}{2}\cdot\frac{l}{N} : l$$

$$\therefore P'_{n-1} = w'\,\frac{n(n-1)}{2N}$$

$$\therefore V_n = w'\,\frac{n(n-1)}{2N} - w' \text{ . - - - - - - - - - - - - - } (2)$$

If the diagonals are iron rods they will then resist the shearing by pulling and equation (1) gives the amount. All verticals will then be struts and the push on the n^{th} when the bridge is <u>not</u> loaded up to n but loaded on n and beyond will be a maximum

The resistance of the right abutment, due to the load that is <u>not</u> on, is downward and we have

$$P'_{n-1} = w'\,\frac{n(n-1)}{2N}$$

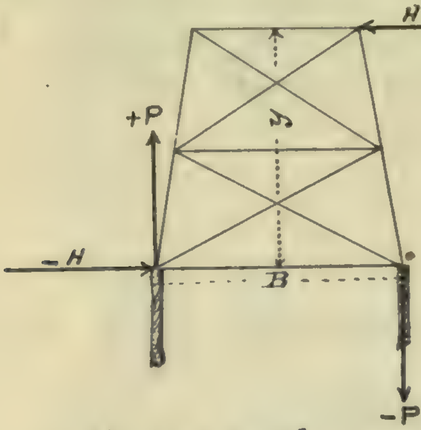
$$\therefore V_n = P'_{n-1} + w'' + w' = w'' + w'\left\{1 + \frac{n(n-1)}{2N}\right\} \text{ - - - - - - - - } (3)$$

p483 See discussion of Iron Bow String Girder p. 562 § 379

§347 The equations of § 346 cannot be accurately applied to this § 347 because the resistance of the abutments is always vertical in the Bow String Girder.

p484 *Timber Piers*

§348



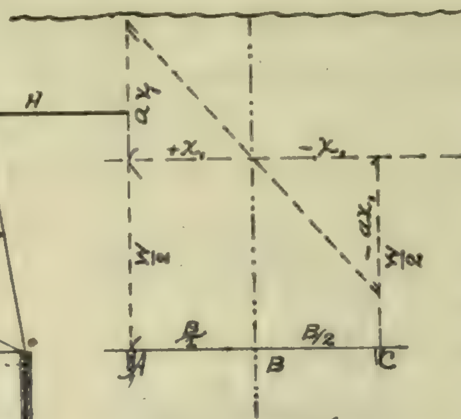


Diagram of Forces

The active moment Hy must equal the passive moment which will be produced by it, consisting of an upward pull on C and a downward push on A

but $P' = ax_1$

$$\therefore Hy = P'B$$

$$\therefore Hy = 2ax_1^2 \quad \therefore a = \frac{Hy}{2x_1^2} = \frac{Hy}{\frac{B^2}{2}} = \frac{2Hy}{B^2}$$

$$\text{Load on A} = \frac{W}{2} + ax_1 = \frac{W}{2} + \frac{2Hy}{B^2}\cdot\frac{B}{2} = \frac{W}{2} + \frac{Hy}{B} \quad \text{----------} \quad (2)$$

$$\text{" " C} = \frac{W}{2} - ax_1 = \frac{W}{2} - \frac{2Hy}{B^2}\cdot\frac{B}{2} = \frac{W}{2} - \frac{Hy}{B}$$

When the load on C is not less than 0 there cannot be negative stress; if it is equal to 0

$$\frac{W}{2} - \frac{Hy}{B} = 0 \quad \therefore B = \frac{2Hy}{W} \quad \text{----------------} \quad (1)$$

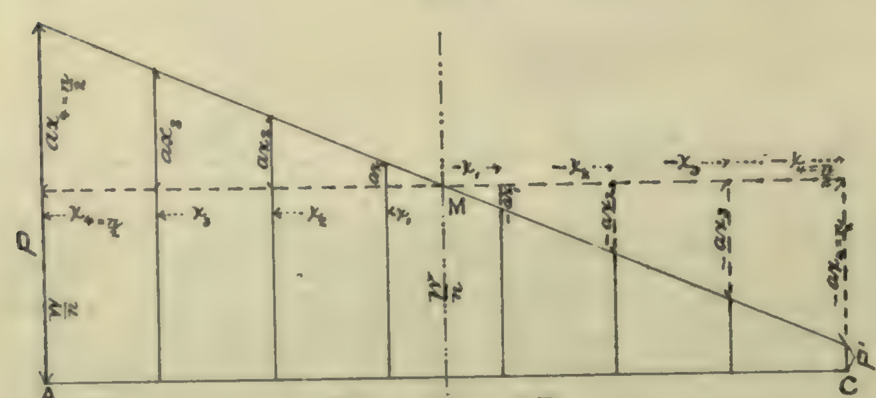


Diagram of Forces

The force will be uniformly distributed $\therefore$

$$Hy = ax_1^2 + ax_2^2 + ax_3^2 \cdots + ax_{n/2} + ax_1 + ax_2 + ax_3 \cdots + ax_{n/2}$$

$$= 2a(x_1^2 + x_2^2 + x_3^2 \cdots\cdots + x_{n/2}^2 \quad \therefore a = \frac{Hy}{2(x_1^2 + x_2^2 + x_3^2 \cdots + x_{n/2}^2}$$

p484 §348 Cont. The load on the post at A

$$= \frac{W}{n} + \frac{Hy x_{n/2}}{2(x_1^2+x_2^2+x_3^2 \dots + x_{n/2}^2)}$$

But $x_{n/2} = MA = \frac{B}{2}$

$$x_{\frac{n}{2}-1} = \frac{B}{2} - \frac{B}{n-1} = \frac{B(n-3)}{2(n-1)} \qquad x_1 = x_{n/2-(\frac{n}{2}-1)} = \frac{B}{2} - \left(\frac{n}{2}-1\right)\frac{B}{n-1} = \frac{B[n-(n-1)]}{2(n-1)}$$

$$x_{\frac{n}{2}-2} = \frac{B}{2} - \frac{2B}{n-1} = \frac{B(n-5)}{2(n-1)} \qquad = \frac{B}{2(n-1)}$$

$$\therefore \frac{W}{n} + a x_{n/2} = \frac{W}{n} + \frac{Hy \cdot \frac{B}{2}}{2\left[\left(\frac{B}{2}\right)^2 + \frac{B^2(n-3)^2}{4(n-1)^2} + \frac{B^2(n-5)^2}{4(n-1)^2} + \dots \frac{B^2}{4(n-1)^2}\right]}$$

$$\therefore \text{Load on } A = \frac{W}{n} + \frac{Hy(n-1)^2}{B[(n-1)^2+(n-3)^2+(n-5)^2+\dots+1]} \quad \text{-------------} (3)$$

$$P' = \text{Load on } C = \frac{W}{n} - \frac{Hy(n-1)^2}{B[(n-1)^2+(n-3)^2+(n-5)^2+\dots+1]} \quad \text{-------------} (4)$$

If we make the condition that there shall be no load on C that is $P'=0$ $\therefore$ $B = \frac{Hy \cdot n(n-1)^2}{W\{(n-1)^2+(n-3)^2+(n-5)^2 \dots +1\}}$ ---------- (5)

in which case, as may be seen either from Equation (3) or from the *Diagram of Forces*, $P = \frac{2W}{n}$ ---------------- (6)

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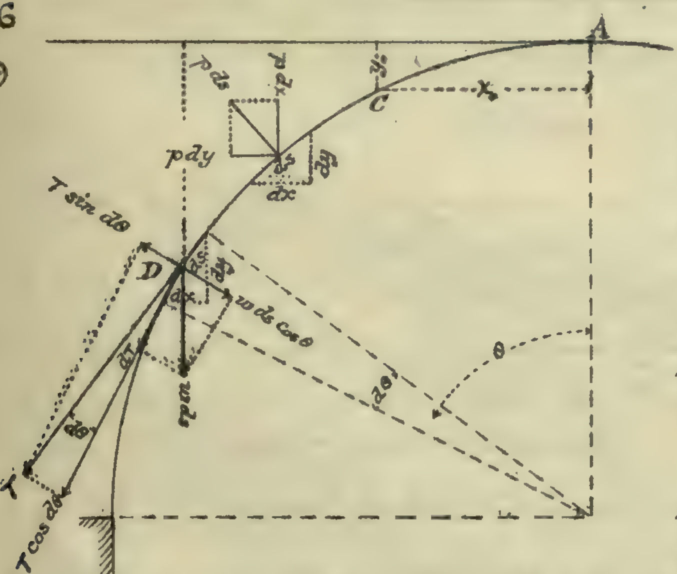
p486 §349



Consider the forces acting on a length of the arch ds. We decompose the weight $w ds$ into two components one radial $= w ds \cos\theta$ = pressure on the centering, and tangential dT which is the increment to the thrust of the rib $= w ds \sin\theta$

Let the thrust at the upper end of $ds = T$ which we decompose into two components one $T\cos d\theta$ in the direction of the rib at the lower end of ds equal $T\cos d\theta$, the other $T \sin d\theta$ normal to the rib at D.

Since $d\theta$ is very small $\cos d\theta = 1$ $\therefore T\cos d\theta = T$ so the same thrust, that presses against the upper end of ds, is transmitted to the lower end; to which must be added the increment which comes from $w ds = w ds \sin\theta = w ds \frac{dy}{ds} = w\,dy = dT$.

p486 The total pressure on the centering for the distance ds

§349 is $p\,ds = w\,ds\cos\theta - T\sin d\theta$; but $\sin d\theta = \frac{ds}{r}$ $\therefore$

Cont. $p\,ds = w\,ds\cos\theta - T\frac{ds}{r}$

$$\therefore\ p = w\cos\theta - \tfrac{1}{r}T$$

To find the value of T we have $T = \int_{y_0}^{y} dT$ but as we saw from the parallelogram of forces at D

$$dT = w\,ds\sin\theta \quad \therefore\ T = \int_{y_0}^{y} w\,ds\sin\theta = \int_{y_0}^{y} w\,dy$$

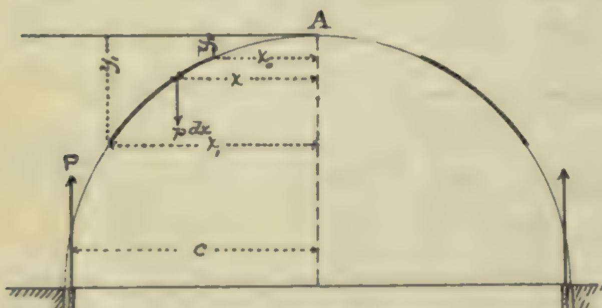
$$\therefore\ p = w\cos\theta - \frac{1}{r}\int_{y_0}^{y} w\,dy \text{ .... (1)}$$

The maximum value of which is $= w\cos\theta$ .... (1 A)

To find the total vertical load we must decompose each $p\,ds$ into its vertical component $p\,dx$, and its horizontal component $p\,dy$, as is done at E. The vertical component is $p\,ds\cos\theta$ but $ds\cos\theta = dx$ $\therefore$ $p\,dx$ = vertical pressure for the distance dx.

$$\therefore\ P = \int_{x_0}^{x} p\,dx \text{ .... (2)}$$

If the load extends to the key stone $x_0 = 0$, and if $x_1 y_1$ is the point where $p = 0$ $\quad P_1 = \int_0^{x_1} p\,dx$ [making $y_0 = 0$] .... (2 A)



If the centering is supported at the ends only then the moment at the centre will be

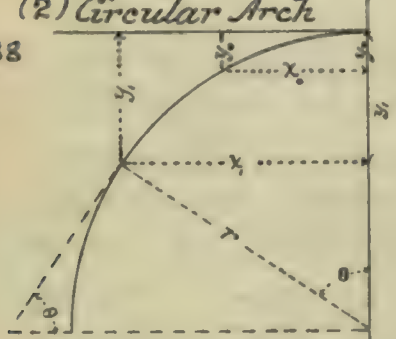
$$M = Pc - \int_{x_0}^{x} p\,x\,dx \text{ ...... (3)}$$

The maximum value of which will occur just before the key-stone is put in.

$$M_1 = P_1 c - \int_0^{x_1} p\,x\,dx \quad (\text{for } y_0 = 0) \text{ .... (3 A)}$$

(2) Circular Arch

p488



$x = r\sin\theta$, $y = r(1-\cos\theta)$

$x_0 = r\sin\theta_0$, $y_0 = r(1-\cos\theta_0)$

$x_1 = r\sin\theta_1$, $y_1 = r(1-\cos\theta_1)$

From Eq. (1)

$$p = w\cos\theta - \frac{1}{r}\int_{y_0}^{y} w\,dy$$

$$= w\cos\theta - \frac{1}{r}\left[w(y-y_0)\right] \ \therefore$$

$$p = w\cos\theta - \frac{w}{r}\left\{r(1-\cos\theta) - r(1-\cos\theta_0)\right\}.$$

$$p = w\left\{2\cos\theta - \cos\theta_0\right\} = w\cdot\frac{r-2y+y_0}{r} \text{ .... (5)}$$

$$w\cos\theta = w\cdot\frac{r-y}{r} \text{ .... (5 A)}$$

From Eq. 2 $\quad P = \int_{x_0}^{x} p\,dx$; but $x = r\sin\theta$ $\therefore$ $dx = r\cos\theta\,d\theta$ $\therefore$

p 488 $p\,dx = wr\,d\theta\left\{2\cos^2\theta - \cos\theta\cos\theta_0\right\}$ .... (a)

§349 $2\cos^2\theta = 1 + \cos 2\theta \;\therefore\; d(\sin 2\theta) = 2\cos 2\theta\, d\theta \;\therefore \int\cos 2\theta\, d\theta = \frac{\sin 2\theta}{2} \;\therefore$

Cont.

$$\int p\,dx = wr\left\{\theta + \frac{\sin 2\theta}{2} - \sin\theta\cos\theta_0\right\} + C$$

$$\sin 2\theta = 2\sin\theta\cos\theta \;\therefore$$

$$\int p\,dx = wr\left\{\theta + \sin\theta\cos\theta - \sin\theta\cos\theta_0\right\} + C$$

$$P = \int_{x_0}^{x} p\,dx = wr\left\{\theta + \sin\theta\cos\theta - \sin\theta\cos\theta_0 - \theta_0 - \sin\theta_0\cos\theta_0 + \sin\theta_0\cos\theta_0\right\}$$

$$= wr\left\{\theta - \theta_0 - \sin\theta\cos\theta_0 + \sin\theta\cos\theta\right\} \;\therefore$$

$$P = wr\left\{\theta - \theta_0 - \sin\theta(\cos\theta_0 - \cos\theta)\right\} \text{ .... (6)}$$

$$= w\left\{s - s_0 - \frac{x}{r}(y - y_0)\right\} \text{ .... (6)}$$

$$P_1 = wr\left\{\theta_1 - \sin\theta_1(1 - \cos\theta_1)\right\} = w\left\{s_1 - \frac{x_1 y_1}{r}\right\} \text{ .... (6 A)}$$

From Eq. (3 A) $\left\{ M_1 = P_1 c - \int_0^{x_1} p\,x\,dx \right\}$

Eq. (a) $p\,dx = wr\,d\theta\left\{2\cos^2\theta - \cos\theta\cos\theta_0\right\}$ and $\cos\theta_0$ here $= 1$

$$x = r\sin\theta \;\therefore\; p\,x\,dx = wr^2(2\cos^2\theta\sin\theta - \cos\theta\sin\theta)\,d\theta$$

$$\therefore M_1 = P_1 c - wr^2\int_0^{x}(2\cos^2\theta\sin\theta - \cos\theta\sin\theta)\,d\theta$$

$$\therefore M_1 = P_1 c - wr^2\left[\tfrac{2}{3}\cos^3\theta + \tfrac{1}{2}\cos^2\theta\right]_0^{x_1}$$

$$= P_1 c - wr^2\left[-\tfrac{2}{3}\cos^3\theta_1 + \tfrac{1}{2}\cos^2\theta_1 + \tfrac{2}{3} - \tfrac{1}{2}\right]$$

$$= P_1 c - wr^2\left[\tfrac{1}{6} + \frac{\cos^2\theta_1}{2} - \frac{2\cos^3\theta_1}{3}\right]$$

Eq. (6 A) $P_1 = w\left(s_1 - \frac{x_1 y_1}{r}\right) \;\therefore M_1 = w\left\{s_1 c - \frac{c\,x_1 y_1}{r} - \frac{r^2}{6} - \frac{(r-y_1)^2}{2} + \frac{2(r-y_1)^3}{3r}\right\}$ (7)

The 1st of equations (8) is obtained by taking the equation of the circle with origin at A $x^2 + y^2 = 2ry \;\therefore\; r = \left(y + \frac{x^2}{y}\right)\cdot\frac{1}{2}$ ...(8)

An approximate value of s, the length of the arch, is given by assuming the length to be the same as that of a parabola. Then $x^2 = 4my$ and $x_1^2 = 4my_1 \;\therefore\; 4m = \frac{x_1^2}{y_1}$

$$\therefore x^2 = \frac{x_1^2}{y_1}y \;\therefore\; 2x\,dx = \frac{x_1^2}{y_1}dy \;\therefore\; \frac{dy}{dx} = \frac{2y_1 x}{x_1^2}$$

From Calculus $\left\{ s = \int\left\{1 + \frac{dy^2}{dx^2}\right\}^{1/2}dx \right\} \;\therefore\; s = \int_0^{x_1}\left[1 + \frac{4y_1^2}{x_1^4}x^2\right]^{1/2}dx$

developing by the binomial theorem

$$s = \int_0^{x_1}\left[1 + \tfrac{1}{2}\frac{4y_1^2}{x_1^4}x^2 - \tfrac{1}{2}\cdot\tfrac{1}{4}\left(\frac{4y_1^2}{x_1^4}\right)^2 x^4 + \text{etc}\right]dx = x_1 + \tfrac{1}{2}\cdot\tfrac{1}{3}\cdot\frac{4y_1^2}{x_1^4}x_1^3 - \tfrac{1}{2}\cdot\tfrac{1}{4}\cdot\tfrac{1}{5}\left(\frac{4y_1^2}{x_1^4}\right)^2 x_1^5 + \text{etc.}$$

$$\therefore\; s = x\left(1 + \frac{2}{3}\cdot\frac{y_1^2}{x_1^2} - \frac{2}{5}\frac{y_1^4}{x_1^4}\right) \text{ nearly .... (8)}$$

p 489
§349 Cont.

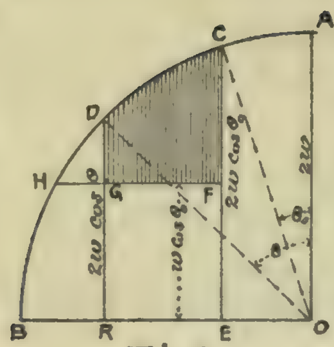


Fig. 3

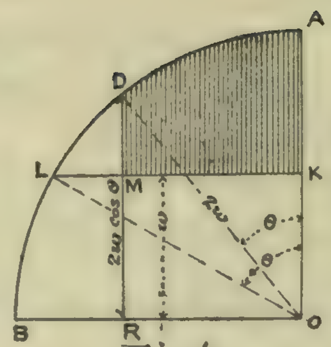


Fig. 4

In Fig. 3 make CF = FE, then DG = DR − GR

$\therefore$ DG = DR − $\frac{1}{2}$CE $\therefore$

DG = $2w\cos\theta - w\cos\theta_0 = p$ ....(5)

This expression for p is general $\therefore$

$P = \int_{x_0}^{x} p\,dx$ = area CFGD

It is apparent that at H the value of $p = 2w\cos\theta - w\cos\theta_0 = 0$

When the arch is built up to the key stone at A as in Fig. 4 bisect DA at K, now $\theta_0 = 0$ and we have

$$DG = 2w\cos\theta - w = p$$

and the load on the centering from A to L the point of no pressure is $P_1 = \int_0^x p\,dx$ = area ALK. while if the load only extends to D we have $P = \int_0^x p\,dx$ = area AKMD

The value of $\theta_1 = \text{arc.}\cos\frac{1}{2} = 60°$

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p 492
§349A The subject of torsion has been discussed on pages 34 and 35 of these Notes, where it was considered for a circular shaft in which case the stress would be directly proportional to the distance from the centre.

Prof. Jas. H Cotterill says (Applied Mechanics p 361) "It is not to be supposed that the strength of a shaft of any section is proportional to the polar moment of inertia of that section" "In non circular sections the stress is generally greatest not at the points farthest away from the centre but more often those which are nearest the centre"

The table on p 36 of these notes gives the strength of a square shaft as 0.8863 times that of a round shaft of the same area.

For a round shaft (p.35 Notes) $\left\{T = \frac{1}{16}\pi f d^3\right\} \therefore M = \frac{1}{16}\pi f d^3$

If a square shaft each side of which is h have the same area $h^2 = \frac{1}{4}\pi d^2 \quad \therefore d = \frac{2h}{\sqrt{\pi}}$

multiplying by 0.8863 the factor given above and replacing

p 492 §349 Cont. d by its value in terms of h we get

$$M = 0.8863\ \frac{\pi}{16}\cdot\frac{2^3}{\pi^{3/2}}\cdot f h^3$$

$$\therefore M = 0.250\, f h^3 \quad \text{------------------} \quad (1)$$

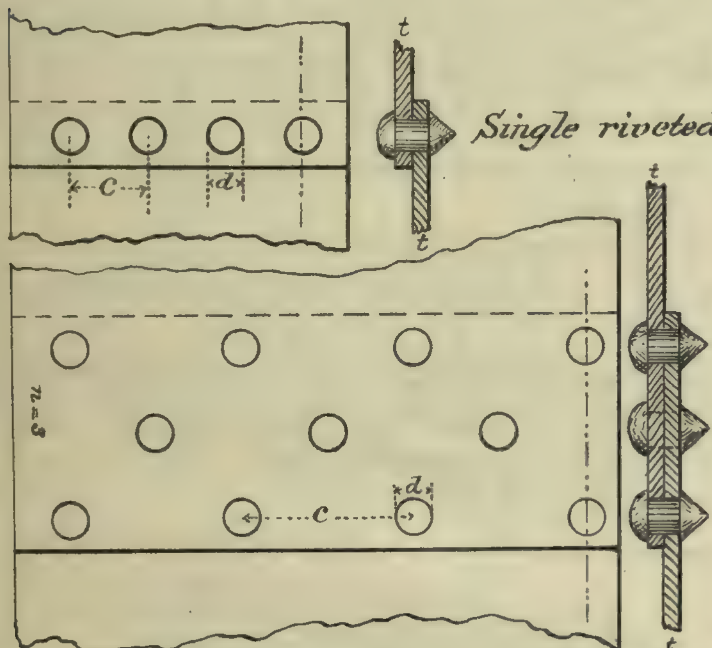
From the equation on p. 35 (Notes) 7th line from top.

$$\left\{ i = \frac{ql}{Cr} \right\} \therefore \theta = \frac{fl}{C \tfrac{1}{2} h} \quad \text{from Eq. 1} \quad f = \frac{M'}{.250 h^3}$$

$$\therefore \theta = \frac{Ml}{.250\, C h^4} \quad \text{------------------} \quad (2)$$

Some of the earlier editions had $M' = 0.280\, f h^3$ for Eq.(1); hence the constant 0.1405 in Eq. 2 as given in the text.

p 517 §363



Single riveted

Triple riveted

$$(c-d)t = \tfrac{1}{4}\pi d^2$$

$$\therefore c = d + \frac{.7854\, d^2}{t}$$

For n rows each row will only have $\frac{1}{n}$th of the stress to bear which the uncut metal between rivets in the outer row has to stand.

$$\therefore \frac{(c-d)t}{n} = .7854\, d^2$$

$$\therefore c = d + \frac{.7854\, d^2 n}{t} \quad \text{.....} \quad (1)$$

$$\frac{c-d}{c} = \frac{\frac{.7854\, n d^2}{t}}{d + \frac{.7854\, n d^2}{t}} = \frac{.7854\, n d}{t + .7854\, n d} \quad \text{......} \quad (2)$$

From Eq. (1) $(c-d)t = .7854\, n d^2 \quad \therefore n = \dfrac{(c-d)t}{.7854\, d^2}$ .... (3)

p 520 §365 Gordon's Formula is $P = \dfrac{fS}{1 + a\frac{l^2}{h^2}}$ [see pp. 67-68 Notes]

where –

- P = total crushing load
- f = crushing load pr. square unit for a short block.
- S = Area of cross-section
- a = an experimental constant

$\left\{ \begin{array}{l} l = \text{length} \\ h = \text{least dimension of Xsection} \end{array} \right.$

p 520
§ 365
Cont.

In applying this formula to cast iron struts of the cross-section of Fig. 1 the dimension h may be measured in the direction h_1 in the figure and the same value of a $(=\frac{1}{500})$ be used as in hollow cast iron cylinders.

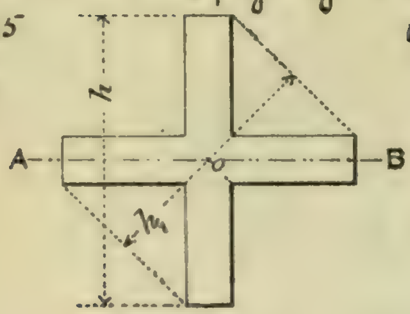


Fig. 1

But it will render the formula more general and Prof. Rankine thinks more "satisfactory" to introduce in place of h [measured as shown in Fig. 1 = h_1] the radius of gyration. The value of this radius is $r^2 = \frac{I}{Mass} = \frac{n' b h^3}{n_1 b h} = \frac{n'}{n_1} h^2$, and its introduction is virtually to separate certain factors n' and n_1 which depend on the form of the cross-section These values are embraced in the experimental factor a and as n' and n_1 are variable with the cross-section they cause a to vary in the same way. When eliminated by introducing r the remaining part of a may be called a' and the formula will then become

$$f = \frac{P}{S}\left(1 + a'\frac{l^2}{r^2}\right) \quad or \quad P = \frac{fS}{1 + a'\frac{l^2}{r^2}} \quad \text{.................} (1)$$

The moment of inertia of an assemblage of rectangles $= \Sigma b h^3$ (where the axis passes through the centre of gravity of each)
∴ In Fig. 1

$$\Sigma b h^3 = I = \tfrac{1}{12} b h^3 + \tfrac{1}{12} h b^3 - \tfrac{1}{12} b^4$$

[The central square being measured twice we subtract its moment of inertia]

$$Area = 2bh - b^2$$

$$\therefore r^2 = \tfrac{1}{24} h^2 + \tfrac{1}{48} b h + etc = \tfrac{1}{24} h^2 \text{ [nearly]}$$

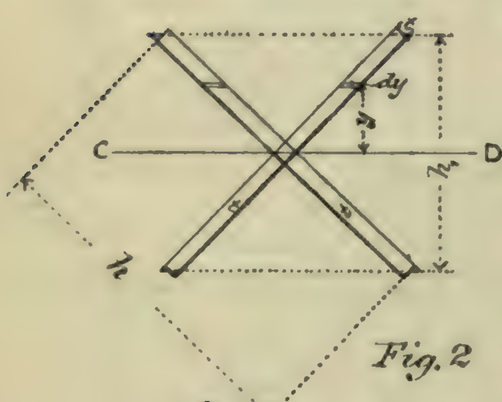


Fig. 2

In this equation A-B is taken as the axis and the approximation is close only when b is very small. If we take C-D as the axis (see Fig. 2) and then assume that counting the central square twice does not materially effect the result; and that the ends may be considered as shaped by terminating them at the dotted lines, which are parallel to the axis.

If two planes be passed parallel to the axis C-D

p. 20 and at a distance apart = dy ; then the length of the
§365 strip which is dy wide is $= 2t\sqrt{2}$

$$\therefore I = 2t\sqrt{2}\int_{-\frac{h_1}{2}}^{+\frac{h_1}{2}} y^2\,dy = 2t\sqrt{2}\left[\frac{h_1^3}{24}+\frac{h_1^3}{24}\right] = 2t\sqrt{2}\,\frac{h_1^3}{12}$$

Area $= 2t\sqrt{2}\times h_1$ $\quad\therefore r^2 = \dfrac{2t\sqrt{2}\,\frac{h_1^3}{12}}{2t\sqrt{2}\,h_1} = \frac{1}{12}h_1^2$ $\quad$ but $h_1 = \dfrac{h}{\sqrt{2}}$

$$\therefore r^2 = \frac{1}{24}h^2 \quad \text{.........} \quad (b)$$

For hollow cylinders [see VIII p. 66 Notes] $I = \frac{1}{64}\pi(h^4 - h'^4)$
Area $= \frac{1}{4}(h^2 - h'^2)$ $\quad\therefore\quad r^2 = \frac{1}{16}(h^2 + h'^2) = \frac{1}{8}h^2$ (nearly) when the cylinder is thin

For hollow cast iron cylinders, [in which case a as deduced from Mr. Hodgkinson's experiments $= \frac{1}{500}$], we can use either

$$P = \frac{fS}{1+\frac{1}{500}\frac{l^2}{h^2}} = P = \frac{fS}{1+\frac{1}{4000}\frac{l^2}{r^2}}$$

For a  we have $P = \dfrac{fS}{1+\frac{3}{500}\frac{l^2}{h}} = P = \dfrac{fS}{1+\frac{1}{4000}\frac{l^2}{r^2}}$
where l and h are the same throughout but the r's vary as shown. In the case of a hollow square strut whose diagonal $= h' =$ diameter of the hollow cylinder, substitute in Case II § 366 for (h) the side of a square its value in terms of the diagonal then $h = \dfrac{h'}{\sqrt{2}}$ and $r^2 = \dfrac{h'^2}{12}$

In the cylinder $r^2 = \dfrac{h'^2}{8}$

$$\text{Hence} \quad P = \frac{fS}{1+\frac{3}{1000}\frac{l^2}{h'^2}} = P = \frac{fS}{1+\frac{1}{4000}\cdot\frac{l^2}{r^2}}$$

In the case where the hollow square has its side $= h =$ diameter of cylinder

$$\text{Square } r^2 = \frac{h^2}{6} \qquad \text{Cylinder } r^2 = \frac{h^2}{8}$$

$$\text{Hence} \quad P = \frac{fS}{1+\frac{3}{2000}\cdot\frac{l^2}{h^2}} = P = \frac{fS}{1+\frac{1}{4000}\cdot\frac{l^2}{r^2}}$$

Note. Baker and others think that it is not worth while to make a change in the formula for hollow cylindrical and hollow square cast iron pillars, so in the next to the last case they would use

$$P = \frac{fS}{1+\frac{1}{500}\cdot\frac{l^2}{\frac{h'^2}{2}}} = \frac{fS}{1+\frac{2}{500}\cdot\frac{l^2}{h'^2}} \qquad \text{And in the last case} \quad P = \frac{fS}{1+\frac{1}{500}\cdot\frac{l^2}{h^2}}$$

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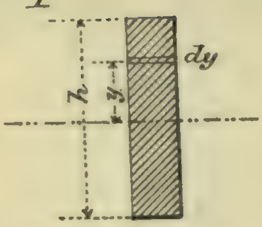
§366

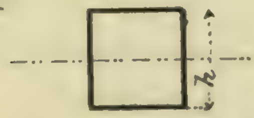
I



$$I = b\int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2\,dy = b\,\tfrac{1}{3}\left(\left(\tfrac{h}{2}\right)^3+\left(\tfrac{h}{2}\right)^3\right) = \frac{bh^3}{12}$$

$$r^2 = \frac{I}{A} = \frac{bh^3}{12\,bh} = \frac{h^2}{12}$$

II



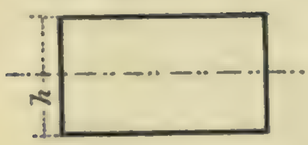
$I = \Sigma I$; I_1 (of the sides) $= \frac{2th^3}{12}$

I_2 (of top and bottom) $= 2\,h\left(\frac{h}{2}\right)^2 t = \frac{1}{2}th^3$

$\therefore I = I_1 + I_2 = \left(\frac{1}{6}+\frac{1}{2}\right)th^3 = \frac{4}{6}th^3$ but Area $= 4ht$

$$\therefore r^2 = \frac{\frac{4}{6}th^3}{4th} = \frac{h^2}{6}$$

III



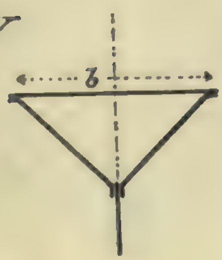
$I = I_1 + I_2$ where I_1 (of ends) $= \frac{1}{6}th^3$

I_2 (" top and bottom) $= 2bt\left(\frac{h}{2}\right)^2 = \frac{bth^2}{2}$

$\therefore I = th^2\left(\frac{h}{6}+\frac{b}{2}\right) = \frac{th^2}{6}(h+3b)$ and $A = 2t(h+b)$

$$\therefore r^2 = \frac{\frac{1}{6}th^2(h+3b)}{2t(h+b)} = \frac{1}{12}h^2\cdot\frac{h+3b}{h+b}$$

IV



Let t = thickness, then a line parallel to the neutral axis cuts the metal for the distance $t + t\sqrt{2} = t(1+\sqrt{2})$

$\therefore I = \frac{1}{12}t(1+\sqrt{2})b^3$ and Area $= bt(1+\sqrt{2})$

$$\therefore r^2 = \frac{b^2}{12}$$

V

See p.66 Notes $I = \frac{\pi h^4}{64}$ Area $= \frac{1}{4}\pi h^2$

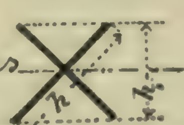
$$\therefore \quad r^2 = \frac{h^2}{16}$$

VI

See p.157 Notes

$$r^2 = \frac{h^2}{8}$$

VII

Nearly same as 

see p.157 Notes $r^2 = \frac{h^2}{24}$

VIII

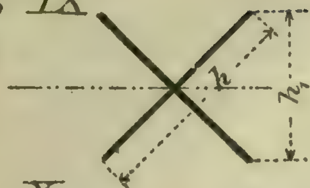
Here the thickness of metal cut by a line or plane parallel to the axis is $t(\sec\theta + \operatorname{cosec}\theta)$

$\therefore I = \frac{t(\sec\theta+\operatorname{cosec}\theta)h_1^3}{12}$ Area $= t(\sec\theta+\operatorname{cosec}\theta)h_1$

$\therefore r^2 = \frac{h_1^2}{12}$ but h_1 is the perpendicular let fall from the right angle on the hypotenuse

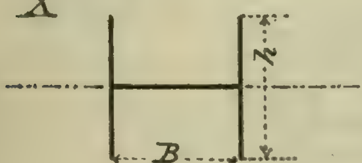
p 523 §366 Cont.

IX



see p. 157 Notes $r^2 = \frac{h^2}{24}$

X

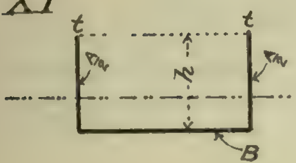


Neglecting the moment of inertia of the cross which we may do when its thickness is very small, its value being $I_1 = Bt^2$,

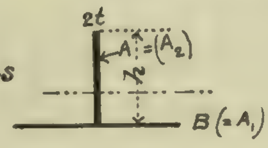
We have $I = \frac{1}{12} A h^2$ Area $= A + B$

$$\therefore r^2 = \frac{h^2}{12} \cdot \frac{A}{A+B}$$

XI



Same as

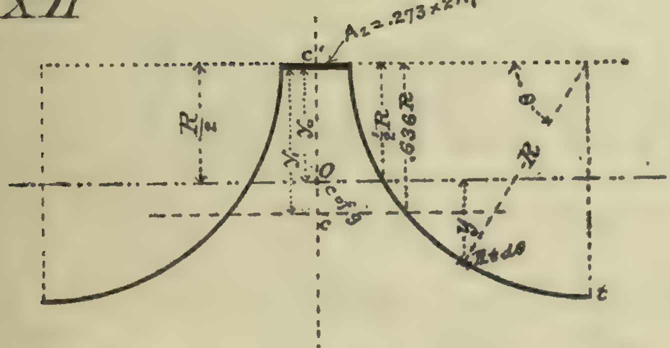


see p 255 Eq. 2 $\left\{ I = h^2\left(\frac{A_2}{12} + \frac{A_1 A_2}{4A}\right) \right\}$

$$\therefore I = h^2\left(\frac{A}{12} + \frac{AB}{4(A+B)}\right)$$

Area $= A + B$ $\therefore r^2 = h^2\left(\frac{A}{12(A+B)} + \frac{AB}{4(A+B)^2}\right)$

XII



t = Thickness

R = Radius

y_1 = c. of g. of quadrants

y_0 = c. of g. " whole rail

A_1 = area of one quadrant

To find y_1 we have

$$A_1 y_1 = \int Rt\, d\theta \times R\sin\theta$$

$$\therefore Rt\frac{\pi}{2} y_1 = R^2 t \int \sin\theta\, d\theta$$

To find y_0

$$2A_1 + A_2 : 2A_1 :: y_1 : y_0$$

$$\therefore y_1 = \frac{2R}{\pi}\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = \frac{2R}{\pi}\left[\cos\theta\right]_0^{\frac{\pi}{2}} = \frac{2R}{\pi} = 0.636$$

$$\pi Rt + 0.273\,\pi Rt : \pi Rt :: 0.636\,R : y_0$$

$$1.273 : 1 :: 0.636\,R : y_0 \qquad \therefore y_0 = \frac{0.636}{1.273} R = \frac{1}{2} R$$

Thus we see that the neutral axis is half way from out to out.

I_1 = Moment of inertia of <u>one</u> quadrant

I_2 = " " " the table whose area is A_2

$$\therefore I = 2I_1 + I_2$$

$$I_2 = 0.273\,\pi Rt \times \left(\tfrac{1}{2}R\right)^2 = \frac{0.273 \times 3.1416}{4} = 0.2145\ R^3 t$$

$I_1 = \int_0^{\frac{\pi}{2}} y^2 (Rt\, d\theta) = Rt\int_0^{\frac{\pi}{2}} y^2\, d\theta$ but, see figure, $y = R\sin\theta - \frac{R}{2}$

$$\therefore I_1 = R^3 t\int_0^{\frac{\pi}{2}} \left(\sin\theta - \tfrac{1}{2}\right)^2 d\theta = R^3 t\int_0^{\frac{\pi}{2}} \left(\sin^2\theta - \sin\theta + \tfrac{1}{4}\right) d\theta$$

p523 but $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\frac{1}{2}+\frac{1}{4} = 3/4$

§366 $\therefore\ I_1 = R^3t\int_0^{\frac{\pi}{2}}\left(\frac{3}{4} - \frac{1}{2}\cos 2\theta - \sin\theta\right)d\theta$

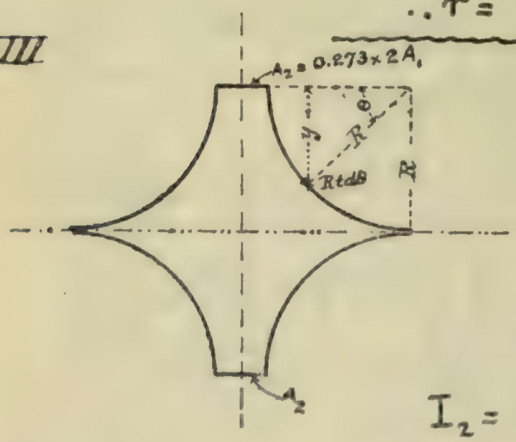
Cont $= R^3t\left[\frac{3}{4}\theta - \frac{1}{4}\sin 2\theta + \cos\theta\right]_0^{\frac{\pi}{2}} = R^3t\left[\frac{3}{4}\frac{\pi}{2} - 0 - (0-0) + (0-1)\right]$

$\therefore I_1 = 0{\cdot}177\ R^3t \quad \therefore 2I_1 = 354\ R^3t$

$\therefore 2I_1 + I_2 = R^3t(0.354 + 0.2145) = 0.5685\ R^3t$

$\therefore r^2 = \dfrac{0.5685\ R^3t}{1.273\ \pi Rt} = \dfrac{R^2}{7}$ nearly

XIII



A_1 = Area of each quadrant

A_2 = " " " table

I_1 = Moment of Inertia each quadrant

I_2 = " " " " table

then $I = 2I_2 + 4I_1$

$I_2 = A_2R^2 = 0.237\pi\ Rt \times R^2 = 0.858\ R^3t$

$2I_2 = 1.716\ R^3t$

$I_1 = \int_0^{\frac{\pi}{2}} y^2\ Rt\,d\theta = Rt\int_0^{\frac{\pi}{2}} y^2\,d\theta$; but $y = R(1-\sin\theta)$

$\therefore I_1 = R^3t\int_0^{\frac{\pi}{2}}(1-\sin\theta)^2 d\theta = R^3t\int_0^{\frac{\pi}{2}}(1 - 2\sin\theta + \sin^2\theta)\,d\theta$

but $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ and $\frac{1}{2} + 1 = \frac{3}{2}$

$\therefore I_1 = R^3t\int_0^{\frac{\pi}{2}}\left(\frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta$

$= R^3t\left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}} = R^3t\left[\frac{3}{2}\cdot\frac{\pi}{2} - 0 + 2(0-1) - \frac{1}{4}(0-0)\right]$

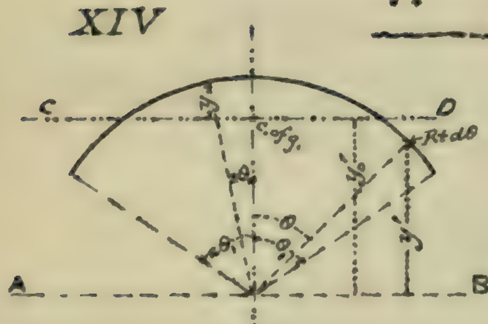
$\therefore I_1 = 0.3564\ R^3t \quad \therefore 4I_1 = 1.4256\ R^3t$

$\therefore I = 4I_1 + 2I_2 = 3.1416\ R^3t$

$\therefore r^2 = \dfrac{I}{A} = \dfrac{3.1416\ R^3t}{2\pi Rt + 0.273\times 2\pi Rt} = \dfrac{R^2}{2.546}$

$\therefore \quad r^2 = 0.393\ R^2$

XIV



To find the distance y_0' of c. of g. from A-B

$2R\theta_1 t$ = Area

$Ay_0' = \int_{-\theta_1}^{+\theta_1} Rt\,d\theta\,y'$

$\therefore 2Rt\theta_1 y_0' = Rt\int_{-\theta_1}^{+\theta_1} R\cos\theta\,d\theta$

$\therefore y_0' = \dfrac{R}{2\theta_1}(\sin\theta_1 + \sin\theta_1) = R\,\dfrac{\sin\theta_1}{\theta_1}$

Taking the axis C-D which passes through the c. of g.

p 524 $I = \int_{-\theta_1}^{+\theta_1} y^2 \, Rt \, d\theta$ but $y = R\cos\theta - \frac{R\sin\theta_1}{\theta_1}$

§366 $\therefore I = R^3 t \int_{-\theta_1}^{+\theta_1} (\cos^2\theta - 2\frac{\sin\theta_1}{\theta_1}\cos\theta + \frac{\sin^2\theta_1}{\theta_1^2}) \, d\theta$

Cont.

but $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ $\therefore I = R^3 t \int_{-\theta_1}^{+\theta_1} [\frac{1}{2} + \frac{1}{2}\cos 2\theta - 2\frac{\sin\theta_1}{\theta_1}\cos\theta + \frac{\sin^2\theta_1}{\theta_1^2}] d\theta$

$$\therefore I = R^3 t \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - 2\frac{\sin\theta_1}{\theta_1}\sin\theta + \frac{\sin^2\theta_1}{\theta_1^2}\theta\right]_{-\theta_1}^{+\theta_1}$$

$$= R^3 t \left[\theta_1 + \tfrac{1}{2}\sin 2\theta_1 - 4\frac{\sin^2\theta_1}{\theta_1} + 2\frac{\sin^2\theta_1}{\theta_1}\right]$$

$\therefore I = R^3 t \left[\theta_1 - \sin\theta_1\cos\theta_1 - 2\frac{\sin^2\theta_1}{\theta_1}\right]$, writing θ for θ_1 to conform with the notation of the text, we have

$$r^2 = \frac{I}{A} = \frac{I}{2Rt\theta_1} = R^2\left[\frac{1}{2} - \frac{\cos\theta\sin\theta}{2\theta} - \frac{\sin^2\theta}{\theta^2}\right]$$

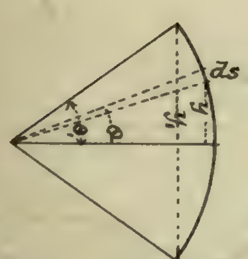
p 527

§369 The moment of resistance in an I beam, the same above and below is (see table, p. 254 I, where $n=6$)

$$M_o = f_a\left(\frac{h' b'^2}{6} + A_1 h'\right) = f_a h'\left(\frac{A_2}{6} + A_1 h'\right) \text{ - - - - - - - - - - } \quad (2)$$

p 528

§370 For an inverted segmental trough subtending the angle 2θ



$$I = \int y^2 ds = \int R^2 \sin^2\theta \cdot R\,d\theta = R^3 \int \sin^2\theta \, d\theta$$

but $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ $\therefore$

$$I = R^3 \int_{-\theta_1}^{+\theta_1} [\tfrac{1}{2}d\theta - \tfrac{1}{2}\cos 2\theta \, d\theta] = R^3\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_{-\theta_1}^{+\theta_1}$$

$$= R^3(\theta_1 + \tfrac{1}{2}\sin 2\theta_1)$$

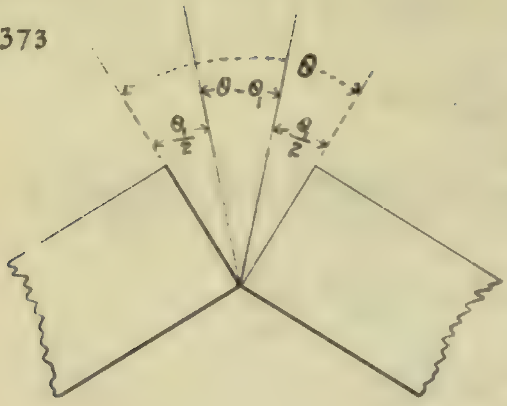
$y_1 = R\sin\theta_1$ and $b = 2y_1 = 2R\sin\theta_1$ $\therefore R = \frac{b}{2\sin\theta_1}$

$r^2 = \frac{I}{A} = \frac{R^3(\theta_1 - \frac{1}{2}\sin 2\theta_1)}{2R\theta_1} = R^2\left(\frac{1}{2} - \frac{1}{4}\frac{\sin 2\theta_1}{\theta_1}\right)$; since we have used θ_1 to mean the same as θ in the text we have

$$r^2 = R^2\left(\frac{1}{2} - \frac{\sin 2\theta}{4\theta}\right) \quad \text{and} \quad R = \frac{b}{2\sin\theta} \quad \therefore$$

$r^2 = \frac{b^2}{8\sin^2\theta}\left(1 - \frac{\sin 2\theta}{2\theta}\right)$; hence the factor by which $\frac{l^2}{b^2}$ must be divided to get $\frac{l^2}{r^2}$ is $\frac{1}{8\sin^2\theta}\left(1 - \frac{\sin 2\theta}{2\theta}\right)$

p 537
§ 373



If to the free beam we introduce a moment at the pier $= M_1 = \frac{2w+w'}{24}$ it will close up the angle θ

If we introduce a less moment $\frac{M_1+M_0}{2} = \frac{w+w'}{16}l^2$ it will change the slope in proportion to the moments, and if we let θ_1 be the angle closed we have

$$\theta : \theta_1 :: \frac{2w+w'}{24}l^2 : \frac{w+w'}{16}l^2 \therefore \theta_1 = \frac{3}{2}\,\frac{w+w'}{2w+w'}\,\theta$$

$\theta - \theta_1 =$ size of the wedge to be inserted

$$\theta - \theta_1 = \left(1 - \frac{3}{2}\,\frac{w+w'}{2w+w'}\right)\theta = \frac{4w+2w'-3w-3w'}{2(2w+w')}\theta = \frac{w+w'}{4w+2w'}\,\theta \quad \text{---} (1)$$

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p.539 Case I See Cor. I Problem II p. 104 Notes § 180
§ 374 Case IV " " I " VI p. 114 Notes § 180

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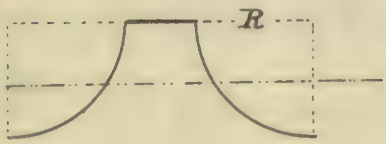
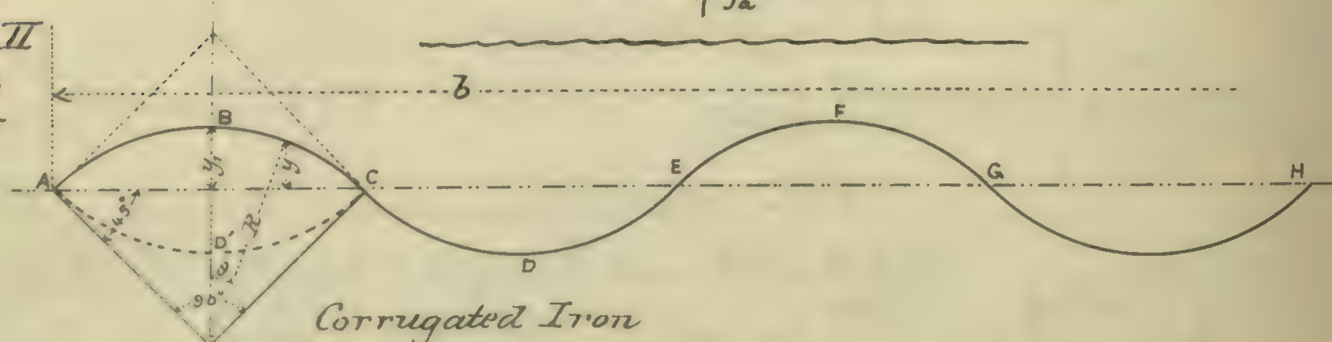
p 543
§ 375



Area Barlow rail $= (1 + 0.273)\pi Rt = 4Rt$.
The neutral axis is shown in these Notes on § 366 to be at the middle of the depth

$$\left.\begin{aligned} M_0 &= \frac{f_a I}{y_1}.\ \text{but } r^2 = \frac{I}{A} = \frac{R^2}{7}\text{, nearly, see table p 523 § 366, and } y_1 = \frac{R}{2} \therefore I = \frac{R^2A}{7} \text{ and } M_0 = \frac{f_a R^2 A}{7\frac{R}{2}} \\ &= \frac{2}{7} f_a RA = \frac{2}{7} f_a R \times \text{area}\ ;\ \text{but area} = 4Rt \therefore \\ M &= \frac{8}{7} f_a t R^2 \end{aligned}\right\} \text{----} (1)$$

II
p 543
§ 375



Corrugated Iron

Let ACE be the neutral axis with the metal crossing it at the angle 45°

If spreading is prevented then the portion ABCDE consisting of a ridge and a furrow is equivalent to a cell ABCD'A, hence we will find the moment of resistance of this cell which we will call M_1

p 543 §275 $M_1 = \frac{fI}{y_1} = \frac{fI}{h/2}$; but $I = 2\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} y^2 (R t_1 \, d\theta)$ where $\begin{cases} R = \text{Radius} \\ t_1 = \text{actual thickness} \end{cases}$

$y_1 = \frac{h}{2} = R(1-\cos 45°) = R(1-\frac{1}{\sqrt{2}}) \quad \therefore R = \frac{h}{2(1-\frac{1}{\sqrt{2}})}$

$$y^2 = R^2\left(\cos\theta - \frac{1}{\sqrt{2}}\right)^2$$

$$\therefore I = 2R^3 t_1 \int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} (\cos^2\theta - \sqrt{2}\cos\theta + \tfrac{1}{2})\, d\theta \quad ; \text{ but } \cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta$$

$$= 2R^3 t_1 \int_{-\pi/4}^{+\pi/4} (1 + \tfrac{1}{2}\cos 2\theta - \sqrt{2}\cos\theta)\, d\theta$$

$$= 2R^3 t_1 \left[\theta + \tfrac{1}{4}\sin 2\theta - \sqrt{2}\sin\theta\right]_{-\pi/4}^{+\pi/4} = 2R^3 t_1 \left[\tfrac{\pi}{4} + \tfrac{\pi}{4} + \tfrac{1}{4}(1+1) - \sqrt{2}\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\right)\right]$$

$$\therefore I = 2R^3 t_1 (\tfrac{\pi}{2} + \tfrac{1}{2} - 2) = .1416\, R^3 t_1$$

$$M_1 = \frac{.1416\, f R^3 t_1}{h/2} = \frac{.1416\, f R^3 t_1}{R(1-\frac{1}{\sqrt{2}})} = \frac{.1416\, f R^2 t_1}{.2929}$$

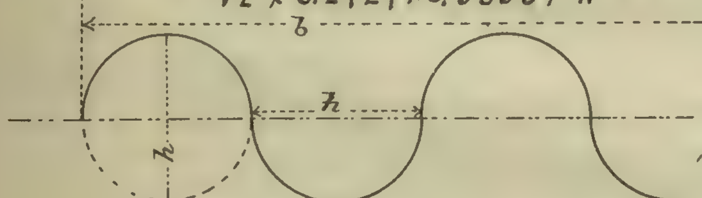
The total number of cells will be $b \div AE$ (where AE is the length of one corrugation) $= \frac{b}{R\sqrt{2}}$

$$\therefore M_0 = \frac{b}{R\sqrt{2}} M_1 = \frac{.1416\, f R b t_1}{\sqrt{2} \times 0.2929} \quad ; \text{ and since } R = \frac{h}{2(1-\frac{1}{\sqrt{2}})} = \frac{h}{.5858}$$

$$M_0 = \frac{.1416\, f h b t_1}{\sqrt{2} \times 0.2929 \times 0.5858}$$

$$\frac{t}{t_1} = \frac{\text{virtual thickness}}{\text{actual thickness}} = \frac{\text{length of arc}}{\text{chord}} = \frac{\frac{\pi}{2}}{\sqrt{2}} = \frac{\pi}{\sqrt{2}} \quad \therefore t_1 = \frac{\sqrt{2}}{\pi} t$$

$$\therefore M_0 = \frac{.1416 \times \sqrt{2}}{\sqrt{2} \times 0.2929 \times 0.5858 \times \pi} f h b t = 0.263\, f h b t = \frac{4}{15.2} f h b t \quad \text{.......} (2)$$



If the surface of the metal crosses the neutral axis at 90° then each cell is a circle $M_1 = f\frac{I}{y_1}$, but $r^2 = \frac{I}{A} = \frac{h^2}{8}$ [see VI on p 523, § 366] $\therefore I = \frac{Ah^2}{8}$

but $A = 2ht$ where t = virtual thickness

$\therefore I = \frac{h^3 t}{4} \quad \therefore M_1 = f\frac{h^3 t}{4 \cdot \frac{h}{2}} = \frac{fh^2 t}{2}$ for each cell, but the number of cells is $\frac{b}{2h}$ $\quad \therefore \quad M_0 = \frac{b}{2h} M_1 = \frac{1}{4} f b h t = .25\, f b h t = \frac{4}{16} f b h t$.

If the crossing be at 60° with the neutral axis

$$M_0 = .259\, f b h t = \frac{4}{15.4} f b h t .$$

If the crossing be at 30° with the neutral axis

$$M_0 = .240\, f b h t = \frac{4}{16.7} f h b t .$$

From which we see that $\frac{4}{15} f h b t$ is a little too large and that $\frac{1}{4}$ is not a poor approximation

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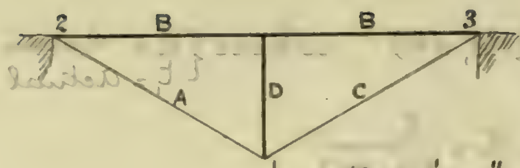
§ 377



$$\left\{ M = \frac{p' I}{y_1} \right\} \quad \left\{ I = n' b h^3, \ y_1 = m' h \right\}$$

$$\left\{ q = \frac{I}{m' h^2 A} \right\}$$ see page 295.

$$p = p' + p'' = \frac{M y_1}{I} + \frac{H}{A} = \frac{M m' h}{q m' h^2 A} + \frac{H}{A}$$

$$\therefore \ p = \frac{1}{A}\left(\frac{M}{q h} + H \right) \text{ - - - - - - - - - - - - - - - - - (1)}$$

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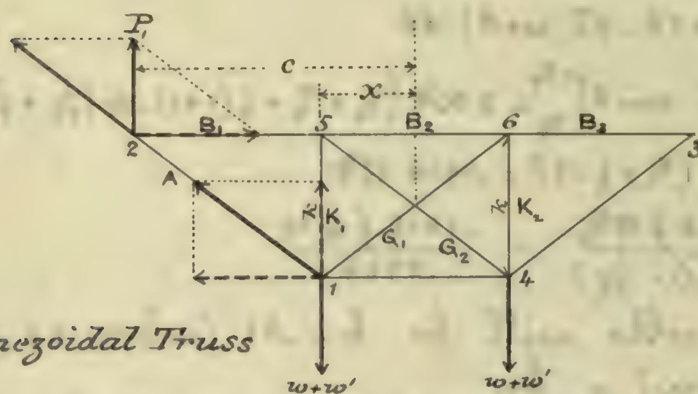
§ 377



Trapezoidal Truss

Diagram of forces shows the action when the travelling load covers the bridge.

The greatest strain on every piece, except the diagonals G_1 and G_2, occurs when both permanent and travelling loads are on.

Let w = permanent load resting at 1 including the weight of

$$\frac{B_1}{2} + \frac{B_2}{2} + \frac{A}{2} + \frac{G_1}{2} + \frac{G_2}{2} + K_1 + \frac{F}{2} = B + G + K + \frac{A+F}{2}$$

Let w' = the greatest travelling load that can come on 5 or 6, and suppose both 5 and 6 loaded.

$w + w' = P_1 = P_2$ = the resistance of the abutments

Then max. $H : w + w' :: c - x : k \quad \therefore \ H = w + w' \dfrac{c-x}{k}$ - - - - - - - - - - (2)

The greatest thrust on each vertical, K_1 or K_2, is

$V = w + w' - (K + \tfrac{1}{2}G)$ - instead of (3 A)

The weight $K + \frac{1}{2}G$ rests at the bottom, 1 and 4, and not at the top points 5 and 6.

The greatest stress in G_1 will occur when 6 is loaded by w' and 5 is unloaded. In this case the resistance of the abutment at 2 is $P_1' : w' :: c - x : 2c \quad \therefore \ P_1' = w' \dfrac{c-x}{2c}$ which would also be the upward thrust in K_1 which would have to be resisted at the point 5 by the two pieces G_1 and B_2

$$\therefore \ S : w' \frac{c-x}{2c} :: \text{length } G_1 : k$$

$$\therefore \ S = w' \frac{c-x}{2ck} \sqrt{4x^2 + k^2} \text{ - - - - - - - - - - - - - - for (3)}$$

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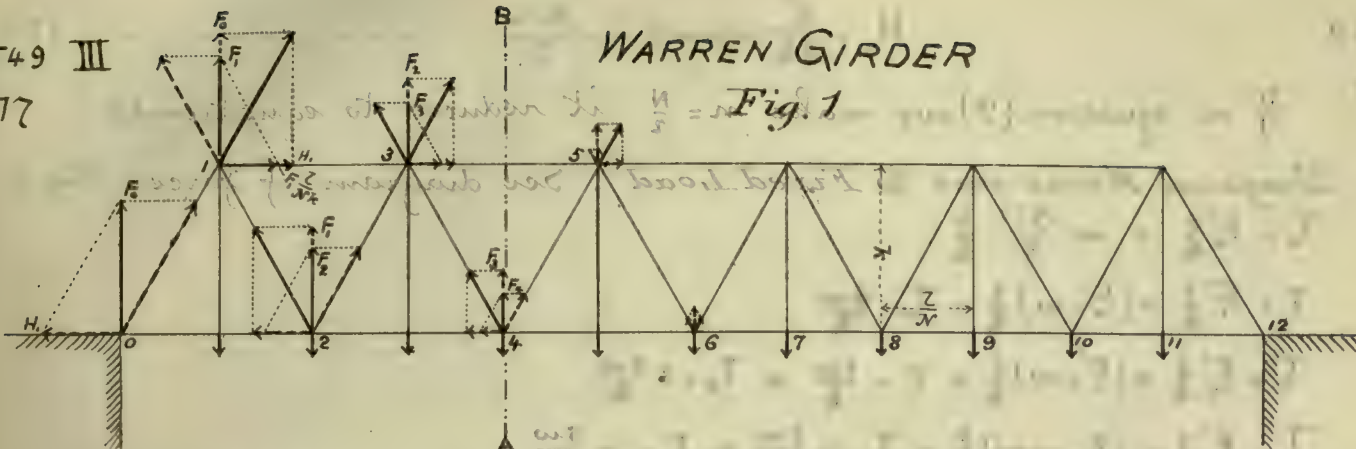
§ 377

WARREN GIRDER

Fig. 1



The above figure shows the forces brought into play when the travelling load covers the entire bridge, in which case each small arrow at the bottom represents $w+w'$. The same figure represents the action of the permanent load if we take the small arrows to represent w.

Beginning at the abutment o the active force at each point is shown by a full line, and the stresses produced by dotted lines.

By inspecting Fig. 1 we see that

$$\left.\begin{aligned}
H_1 &= F_0 \frac{l}{Nk} = (w+w')\frac{N-1}{2}\cdot\frac{l}{Nk} \\
H_2 &= H_1 + F_1 \frac{l}{Nk} \\
H_3 &= H_1 + F_1 \frac{l}{Nk} + F_2 \frac{l}{Nk} = H_2 + F_2 \frac{l}{Nk} \\
H_4 &= H_1 + F_1 \frac{l}{Nk} + F_2 \frac{l}{Nk} + F_3 \frac{l}{Nk} = H_3 + F_3 \frac{l}{Nk} \\
\therefore H_n &= H_{n-1} + F_{n-1} \frac{l}{Nk}
\end{aligned}\right\} \quad \text{- - - - - - - - - - - - - - -} \quad (7)$$

For the middle of the bridge $M_o = mWl$ and from the table p. 247 § 161 case VIII p. 70 of these Notes, when N is even $m = \frac{N}{8(N-1)}$; and $W = (N-1)(w+w')$

$$\therefore M_o = \frac{Nl}{8(N-1)}(N-1)(w+w') = \frac{Nl}{8}(w+w') = H_{\frac{N}{2}} k$$

$$\therefore H_{\frac{N}{2}} = \frac{Nl}{8k}(w+w') \text{ - - - - - - - - - - - - - - } (8)$$

To find H_n see p 247 § 161 $\quad M = \frac{n(N-n)}{2N} l \, (w+w') = H_n k$

$$\therefore H_n = \frac{l}{Nk}(w+w') \frac{n(N-n)}{2} \text{ - - - - - - - - - - - - - - - - } (9)$$

This equation may also be found by taking a vertical section AB through $n\,(=4)$

$$H_n k = F_0 n \frac{l}{N} - (n-1)(w+w')\, n \frac{l}{2N} = (w+w')\left[\frac{N-1}{2} n \frac{l}{N} - \left(\frac{n-1}{2}\right) n \frac{l}{N}\right]$$

p 549 $\quad \therefore \; H_n = \frac{l}{Nk}(w+w')\, n \cdot \frac{N-n}{2}$ - - - - - - - - - - - - - (9)

§ 377 Cont. If in equation (9) we make $n = \frac{N}{2}$ it reduces to equation (8)

Diagonal stress due to Fixed Load - See diagram of forces Fig. 1

$T_0 = F_0' \frac{s}{k} = w \frac{N-1}{2} \cdot \frac{s}{k}$

$T_1 = F_1' \frac{s}{k} = (F_0' - w)\frac{s}{k} = T_0 - \frac{sw}{k}$

$T_2 = F_2' \frac{s}{k} = (F_1' - w)\frac{s}{k} = T_1 - \frac{sw}{k} = T_0 - 2\frac{sw}{k}$

$T_n = F_n' \frac{s}{k} = (F_{n-1} - w)\frac{s}{k} = T_{n-1} - \frac{sw}{k} = T_0 - n\frac{sw}{k}$

For the diagonals adjoining the middle joint of the girder

$n = \frac{N}{2} - 1 \; \therefore \; T_{\frac{N}{2}-1} = T_0 - \left(\frac{N}{2} - 1\right)\frac{sw}{k} = \frac{N-1}{2} \cdot \frac{sw}{k} - \frac{N-2}{2} \cdot \frac{sw}{k} = \frac{sw}{2k}$

For the diagonal next beyond the middle $\quad n = \frac{N}{2}$

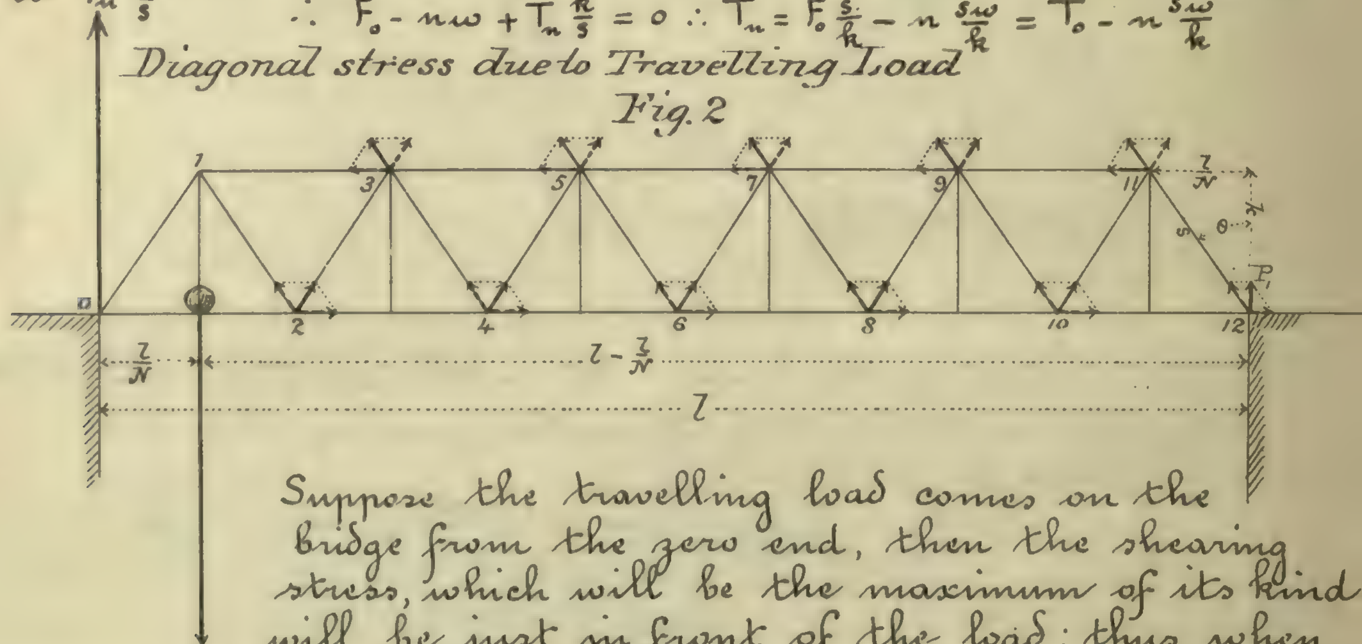
$$\therefore \; T_{\frac{N}{2}} = T_0 - \frac{N}{2} \cdot \frac{sw}{k} = \frac{N-1}{2} \cdot \frac{sw}{k} - \frac{N}{2}\frac{sw}{k} = -\frac{sw}{2k}$$

To get T_n by the method of sections - Suppose the section to be taken on AB and cutting the n^{th} diagonal; then since $\Sigma X = 0$ and the vertical component of the thrust in this diagonal will be $T_n \frac{k}{s}$

$$\therefore \; F_0 - nw + T_n \frac{k}{s} = 0 \; \therefore \; T_n = F_0 \frac{s}{k} - n\frac{sw}{k} = T_0 - n\frac{sw}{k}$$

Diagonal stress due to Travelling Load

Fig. 2



Suppose the travelling load comes on the bridge from the zero end, then the shearing stress, which will be the maximum of its kind will be just in front of the load; thus, when 1 is loaded, we have the max. stress, of its kind (compression), in the 1st diagonal (from 1 to 2) = S_1

When the load is on 1 and 2 we have max. stress of its kind (tension) which can come on diagonal # 2 and this will be the sum of the stresses which come from w' at 1 and w' at 2 and = S_2

p 553 Let P_1 = resistance of abutment at 12 when load is at 1 only
§377 P_2 = resistance of same when the load w' is at 2 only and so
cont. for P_3, P_4 &c. then for load at

1 only $\quad l : \frac{l}{N} :: w' : P_1 = \frac{w'}{N} \therefore \quad S_1 = P_1 \sec\theta = \frac{w'}{N} \cdot \frac{s}{k} = \frac{w's}{Nk}$

2 " $\quad l : \frac{2l}{N} :: w' : P_2 = \frac{2w'}{N} \therefore \quad S_2 = (P_1 + P_2)\sec\theta = \frac{(1+2)w'}{N} \cdot \frac{s}{k} = \frac{3w'}{Nk}$

3 " $\quad l : \frac{3l}{N} :: w' : P_3 = \frac{3w'}{N} \therefore \quad S_3 = (P_1 + P_2 + P_3)\sec\theta = \frac{(1+2+3)w'}{N} \cdot \frac{s}{k} = \frac{6w'}{Nk}$

n " $\quad l : \frac{nl}{N} :: w' : P_n = \frac{nw'}{N} \therefore \quad S_n = (P_1 + P_2 + \cdots P_n)\sec\theta = \frac{(1+2+3+\cdots n)ws}{Nk}$

These stresses will be different in kind from those produced by the permanent load, until the centre of the bridge is reached, beyond which point they will be the same in kind as those caused by the permanent load.

The complete table for a bridge of 12 panels as shown in Fig's. 1 and 2 arranged as recommended in the text is

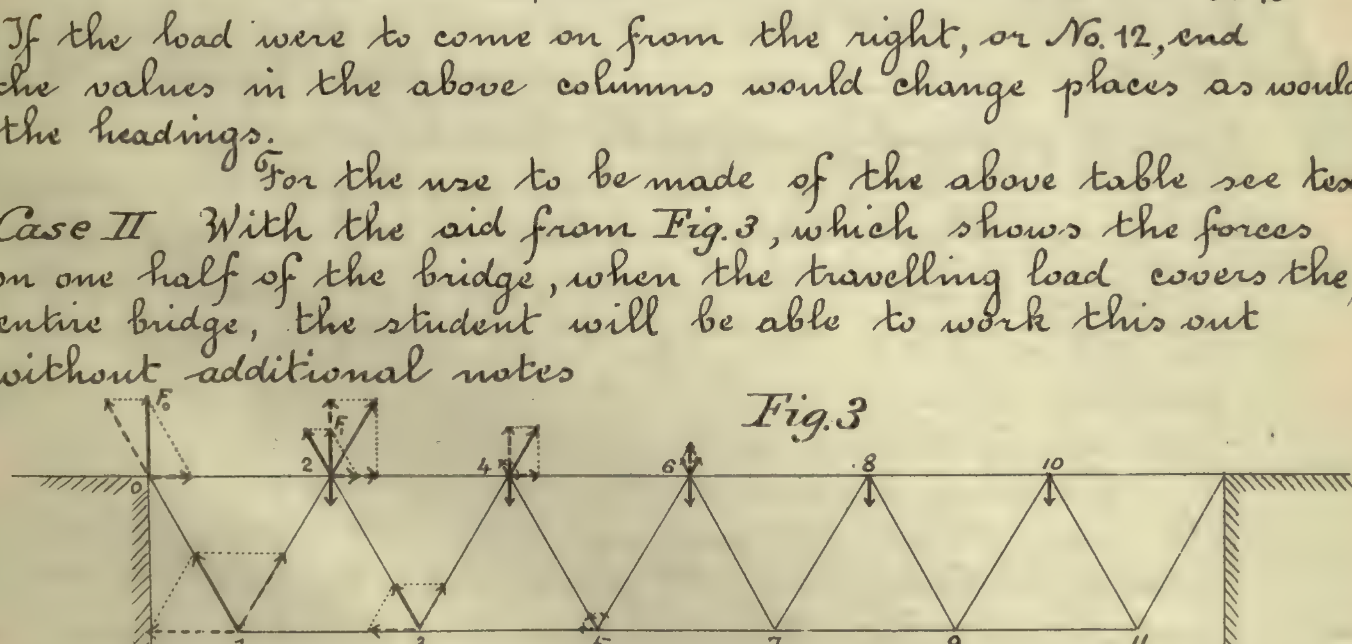
Unlike Permanent Load	*Like Permanent Load*
$S_0 = 0$	$S_{N-1} = S_{11} = (1+2+3+\cdots\cdots 11)\, w'\frac{s}{Nk} = \frac{66\,w's}{Nk}$
$S_1 = w'\frac{s}{Nk}$	$S_{N-2} = S_{10} = (1+2+3+\cdots\cdots 10)\, w'\frac{s}{Nk} = 55\,\frac{w's}{Nk}$
$S_2 = (1+2)\,w'\frac{s}{Nk} = 3\,w'\frac{s}{Nk}$	$S_{N-3} = S_9 = (1+2+3+\cdots\cdots 9)\, w'\frac{s}{Nk} = 45\,\frac{w's}{Nk}$
$S_3 = (1+2+3)\,w'\frac{s}{Nk} = 6\,w'\frac{s}{Nk}$	$S_{N-4} = S_8 = (1+2+3+\cdots\cdots 8)\, w'\frac{s}{Nk} = 36\,\frac{w's}{Nk}$
$S_4 = (1+2+3+4)\,w'\frac{s}{Nk} = 10\,w'\frac{s}{Nk}$	$S_{N-5} = S_7 = (1+2+3+\cdots\cdots+7)\, w'\frac{s}{Nk} = 28\,\frac{w's}{Nk}$
$S_5 = (1+2+3+4+5)\,w'\frac{s}{Nk} = 15\,w'\frac{s}{Nk}$	$S_{N-6} = S_6 = (1+2+3+4+5+6)\, w'\frac{s}{Nk} = 21\,\frac{w's}{Nk}$

If the load were to come on from the right, or No. 12, end the values in the above columns would change places as would the headings.

For the use to be made of the above table see text.

Case II With the aid from Fig. 3, which shows the forces on one half of the bridge, when the travelling load covers the entire bridge, the student will be able to work this out without additional notes

Fig. 3



p562 *Case I*
§379

Fig 1

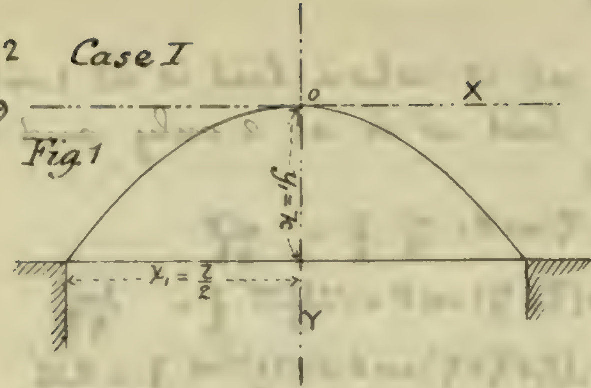


Fig. 2

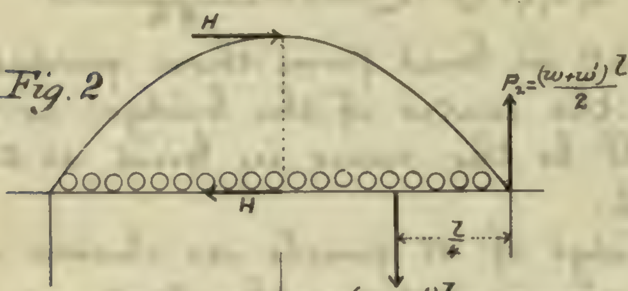
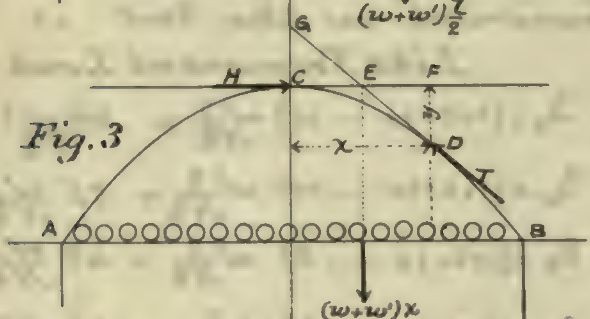


Fig. 3



It has been shown, p. 188 §125. that a parabola is the right form for a suspension bridge uniformly loaded horizontally. Changing the word tension to compression the equations of that article apply to this.

$x^2 = 2py$ is the equation to the curve, substituting the coördinates of the abutment

$$x_1^2 = 2py_1 \quad \therefore \quad 2p = \frac{x_1^2}{y_1} = \frac{l^2}{4k}$$

and we have $x^2 = \frac{l^2}{4k}$.

To find the thrust at the crown and tension in the tie, which are the same, we have the equation of moments, see Fig 2,

$$Hk = \frac{w+w'}{2} . l . \frac{l}{4} \quad \therefore$$

$$H = \frac{(w+w')l^2}{8k} \text{ .... } (1)$$

To find the thrust at any point as D, the three forces H, T, and $(w+w')x$ must be in equilibrium and will therefore be proportional to the triangle EFD

$$\therefore T = \sqrt{H^2 + (w+w')^2 x^2}$$

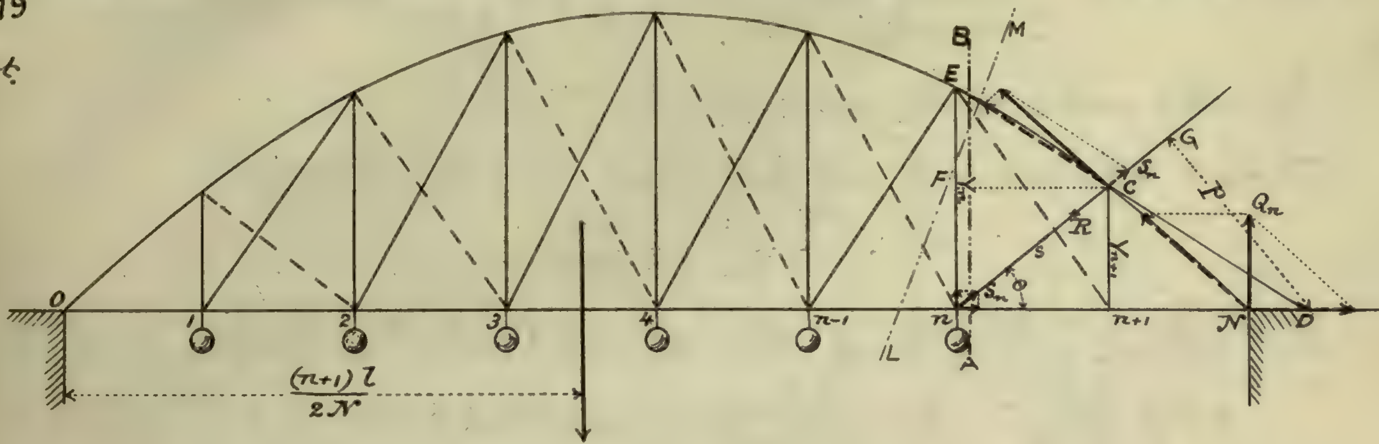
If the supporting rods are not so close together, that the load is practically uniformly distributed, then instead of the parabolic bow we will have a polygon the vertices being, sensibly, in the parabola

Rolling Load. Supposing the load to come on from the left or O end (see Fig. 4, on next page) then the greatest thrust in the n^{th} vertical occurs when the load extends up to and includes the point $n-1$

Rankine's expression (4) can be gotten by supposing the max. thrust to come on the n^{th} vertical when the loading includes the n^{th} joint; and by making the further assumption that the shearing force at and beyond n is the same as the resistance of the right abutment.

p564 §379 Cont.

Fig. 4 Showing forces on right end when load comes up to and includes n



Both these assumptions are in error as may be seen by inspecting Fig. 4 from which it is obvious, first that if a load rest at n it will diminish the thrust in the n^{th} vertical, second that a large part of the resistance of the abutment, $= Q_n$, goes into the thrust along the bow, as a vertical component, since the bow is not horizontal as the upper boom of the bridges formerly discussed is.

To get Rankine's expressions

$$Q_n : n w' \frac{l}{N} :: (n+1)\frac{l}{2N} : l \quad \therefore \quad Q_n = \frac{n(n+1)}{2N} \cdot w' \frac{l}{N} \text{ ------- (a)}$$

If we subtract from this, the part of the permanent load which rests at n, we hav for the compression in the n^{th} vertical

$$\frac{n(n+1)}{2N} \cdot w' \frac{l}{N} - w'' \frac{l}{N} = \frac{l}{N}\left(w' \cdot \frac{n(n+1)}{2N} - w''\right) \text{ ----------- (4)}$$

The thrust in this vertical is caused by a pull along the n^{th} diagonal so if we multiply eq. (a) by $\operatorname{cosec}\theta$ we have

$$\frac{n(n+1)}{2N} \cdot w' \frac{l}{N} \cdot \operatorname{cosec}\theta = \frac{n(n+1)}{2N} \cdot w' \frac{l}{N} \cdot \frac{s}{y} = \frac{w' l s}{y} \cdot \frac{n(n+1)}{2N^2} \text{ ----- (5)}$$

The extent to which this is in error may be seen from Fig. 4 when we notice that the force S_n is very far short of $Q_n \operatorname{cosec}\theta$, which latter would be represented by the line nR.

To find S_n = Greatest stress in n^{th} diagonal, nC.

The load must be as in Fig. 4. Cut by a plane A-B. Consider the bow a polygon, which it either is actually, or virtually; produce EC to D, from D let fall DG, a perpendicular on nC produced.

Take moments about D; Two of the pieces cut pass

p 564 through the point D and therefore the forces acting along
§ 379 them have no lever arms. The equation of moments is
Cont

$$S_n \times p = Q_n \times \mathcal{N}D \quad \therefore \quad S_n = \frac{Q_n \times ND}{p}$$

For the value of Q_n see Eq.(a)

To find $\mathcal{N}D = nD - nN$ we have from similar triangles

$$EF : FC :: En : nD$$

$$\therefore nD = \frac{FC \times En}{EF}$$

Let $nE = Y_n$ = length of the n^{th} vertical; Y_{n+1} = length $(n+1)^{th}$ &c.

$$nD = \frac{\frac{l}{N} \times Y_n}{Y_n - Y_{n+1}} = \frac{l\,Y_n}{\mathcal{N}(Y_n - Y_{n+1})} \quad \therefore \mathcal{N}D = nD - (\mathcal{N}-n)\frac{l}{\mathcal{N}} = \frac{l}{N} \cdot \frac{Y_n}{Y_n - Y_{n+1}} - (N-n)\frac{l}{N}$$

To find p, we have from the triangle nGD

$$p = nD \times \sin\theta = nD\,\frac{Y_{n+1}}{S} = \frac{l}{\mathcal{N}} \cdot \frac{Y_n}{Y_n - Y_{n+1}} \cdot \frac{Y_{n+1}}{s}$$

$$\therefore S_n = \frac{n(n+1)}{2N} \cdot w' \frac{l}{N} \times \left[\frac{l}{N} \cdot \frac{Y_n}{Y_n - Y_{n+1}} - (N-n)\frac{l}{N}\right] \times \frac{1}{\frac{l}{N} \cdot \frac{Y_n}{Y_n - Y_{n+1}} \cdot \frac{Y_{n+1}}{s}}$$

$$\therefore S_n = w' \frac{n(n+1)}{2\mathcal{N}} \cdot \frac{(Y_n - Y_{n+1})\,s}{Y_n Y_{n+1}} \cdot \frac{l}{N}\left[\frac{Y_n}{Y_n - Y_{n+1}} - (N-n)\right] \text{------------- (5A)}$$

To find the greatest thrust in the n^{th} vertical, the load must extend up to, and include, the $n-1$ joint, but not include the n^{th}. Cut by any plane, as LN, the bow, tie and vertical, and take the moments about D as before.

$$V'_n \times nD = Q_{n-1} \times ND \quad \therefore V'_n = \frac{Q_{n-1} \times ND}{nD}$$

but,

$$Q_{n-1} : (n-1)\,w'\frac{l}{N} :: n\frac{l}{2N} : l \quad \therefore Q_n = \frac{n(n-1)}{2N} \cdot w'\frac{l}{N}$$

for values of ND and nD see above:

$$\therefore V'_n = w' \frac{n(n-1)}{2N} \cdot \frac{l}{N} \times \left[\frac{l}{N} \cdot \frac{Y_n}{Y_n - Y_{n+1}} - (N-n)\frac{l}{N}\right] \times \frac{1}{\frac{l\,Y_n}{N(Y_n - Y_{n+1})}}$$

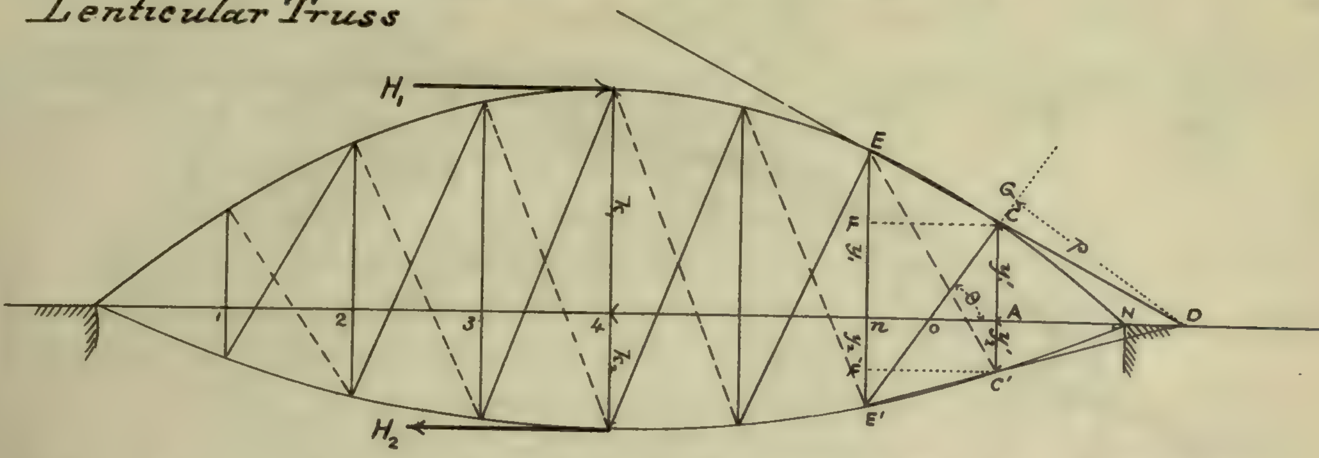
$$V_n = V'_n - w''\frac{l}{N} = w' \frac{n(n-1)\,l}{2N^2} \cdot \left[\frac{Y_n}{Y_n - Y_{n+1}} - (N-n)\right] \frac{Y_n - Y_{n+1}}{Y_n} - w''\frac{l}{N} \text{--------- (4A)}$$

The diagonals shown by broken lines would come into use when the load came on the bridge from the right hand end.

This bridge gives a very economical distribution of materials but the shop work is not so easy, and it is more difficult to brace against wind pressures than the Warren or Whipple trusses.

WHIPPLE TRUSS

p564 *Case III Bowstring Suspension Bridge or*
§379 *Lenticular Truss*



Let w_1 = weight per linear foot, horizontally, of the bow
" w_2 = " " " " " " " tie

Then from Eq. 1 $\quad H_1 = \frac{w_1 l^2}{8k_1} = \frac{w_2 l^2}{8k_2} = H_2 \quad \therefore k_1 : k_2 : k_1 + k_2 :: w_1 : w_2 : w_1 + w_2$

Since this is the case the lower parabola will be a parallel projection of the upper. The point D will be found as before and can be used as centre of moments.

In this case $\theta = \text{arc.}\cos\frac{FC}{CE'} = \text{arc.}\cos\frac{\frac{l}{N}}{s} = \text{arc.}\cos\frac{l}{Ns}$

$p = OD\sin\theta$; but $OD = nD - nO$, and $nO = y_2\cot\theta$

$$p = (nD - y_2\cot\theta)\sin\theta$$

The calculations for this case are similar to those for Case I. It is evident that the horizontal, thrust in the bow, and tension in the tie, must remain equal; so that the amount of the rolling load supported by each will be in the same ratio as the permanent loads or

$$w_1' : w_2' :: w_1 : w_2 :: k_1 : k_2$$

This bridge may in a sense be called ideal, as giving the greatest sustaining power for the amount of material used, but it lacks simplicity of construction and the tie if allowed to come below the roadway is liable to interfere with the water way or roadway underneath.

p567 *Case I Hinged both at crown and springing*
§380 $\quad H_0 k = \frac{1}{8}Wl = \frac{wc^2}{2} \quad \therefore H_0 = \frac{wc^2}{2k}$ -------(1)

$H_1 k = \frac{w'c^2}{2} \quad \therefore H_1 = \frac{w'c^2}{2k}$ -------(2)

$T = Af' = (H_0 + H_1)\sec i \quad \therefore A = \frac{(H_0 + H_1)\sec i}{f'}$ ----- (3)

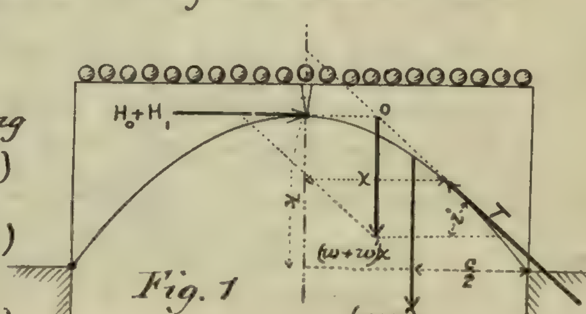
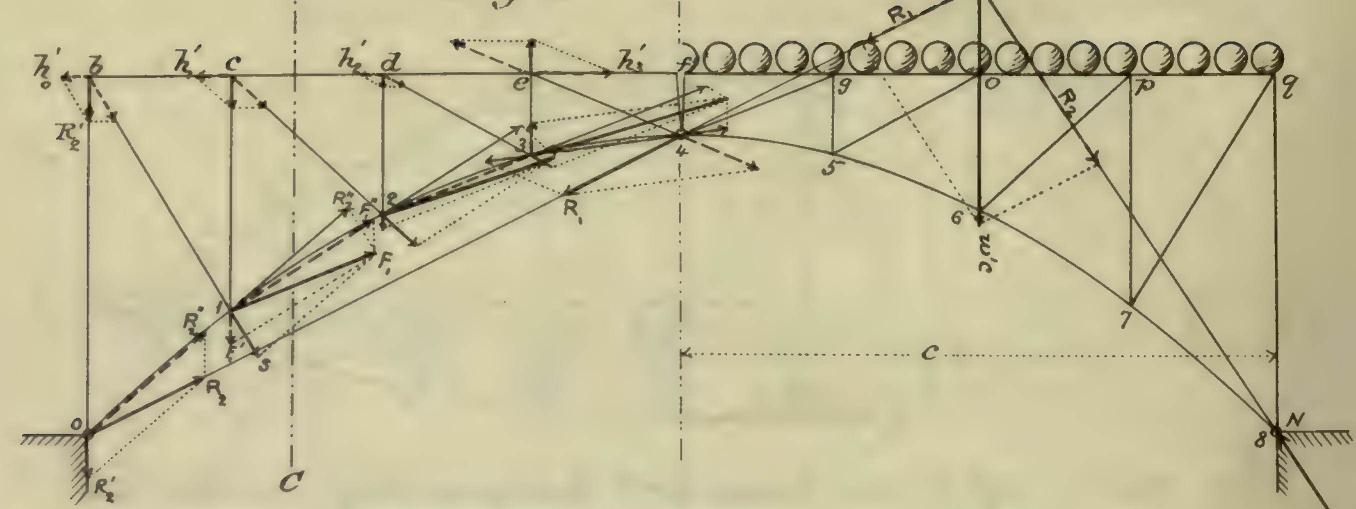


Fig. 1

p.568
§380

Fig 2



Equation 4 can only be gotten by supposing the loaded half to produce the same horizontal thrust, at the crown, that it would do were the other half loaded.

The thrust at the crown is not horizontal in this case but must be in the direction of the line joining the hinge at the springing with the one at the crown. This determines the direction of resistance of the abutment which, as will be seen in Fig. 2, passes through the two hinges; prolong this line to A, where it intersects the line through the c. of g. of the rolling load, take on this vertical line the distance w'c, draw AN which gives the direction R_1, complete the parallelogram as is done in the figure and we have the magnituds as well as the directions of R_1 and R_2

Fig. 2 shows all the forces acting at the joints, and along the several pieces, of the unloaded half, due to the travelling load.

Beginning at the abutment 0, R_2 is decomposed into components R_2' and R_2''; at the joint B, R_2' becomes the active force, which is decomposed into h_0' and S_0; at 1 the forces R_2'' and S_0 are both active forces which are equivalent to their resultant F_1, which in turn must be decomposed in the direction of 1-2 and 1-c into F_1' and F_1''; and so for the other points throughout the bridge.

It will be seen that the resultant thrust along the piece 3-4, at the point 3, just balances the thrust

p568 at the other end which comes from resolving R_2, which
§380 acts at the crown, in the direction of the pieces 4-3 and
Cont. 4-e. It will also be seen that the tension along the piece 4-e is produced by an equal force at each end, the one gotten by starting at the abutment, the other at the crown. Notice that $h'_3 = h'_2 + h'_1 + h'_0$

In a similar manner we could get the forces which act on the loaded half, In this case we would represent the action of the load, by considering $\frac{w'}{2}.\frac{l}{N}$ acting at f and q, and $w'.\frac{l}{N}$ acting at g and the other points.

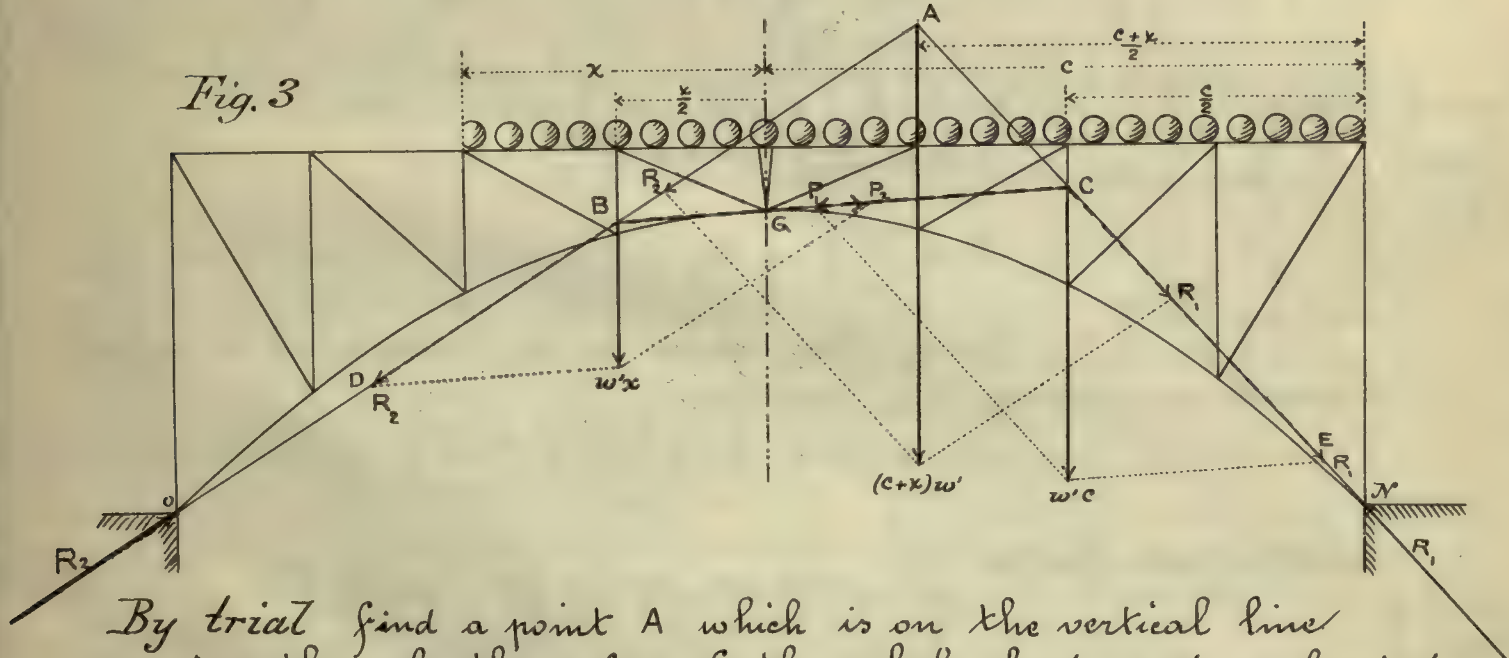
The magnitude of the forces represented in Fig. 2 could be calculated analytically, following the graphical construction.

The method of sections may be used to advantage by taking planes as B---C cutting any three pieces; the only external force acting is R_2

If less than half of the bridge is loaded we proceed in a manner, so nearly identical with the above, that a special figure to illustrate this is not needed.

If more than half the bridge is loaded we proceed as shown in Fig. 3

Fig. 3



By trial find a point A which is on the vertical line passing through the c. of g. of the whole load, and so located that when the parallelograms of force are constructed as shown in Fig. 3 the forces P_1 and P_2 will lie on one and the same straight line, BC, passing through G, the hinge at the crown. Also $P_1 = P_2$

p568 Having found the direction and magnitude of the resistances,
§380 R_1 and R_2, of the abutments, we can find the stress on each
Cont. piece, either by the method of sections, or as was done in the case illustrated in Fig. 2, by constructing parallelograms of forces at each joint.

---

p571 The greatest intensity of stress on a column is found
§381 by taking the algebraic sum of the stress due to the direct component, $= p'$, and that due to the bending moment $= p''$

$\left\{ M = \frac{p'' I}{m' h} \right\} \therefore p'' = \frac{M m' h}{I}$, in this case $m' = \frac{1}{2}$, $h = d$, $M = HY$.

and from table p523 VI $\left\{ r^2 = \frac{I}{A} = \frac{d^2}{8} \right\} \therefore I = A . \frac{d^2}{8}$

$$\therefore p'' = \frac{HY \frac{d}{2}}{A \frac{d^2}{8}} = \frac{1}{A} \cdot \frac{4HY}{d} \text{ ; also } p' = \frac{P}{A} \quad \therefore$$

$$p_1 = p'' + p' = \frac{1}{A}\left\{ \frac{4HY}{d} + P \right\} \text{ - - - - - - - - - - - - - - - } (1)$$

$$p_2 = p'' - p' = \frac{1}{A}\left\{ \frac{4HY}{d} - P \right\} \text{ - - - - - - - - - - - - - - - } (1)$$

In order that there may be no tension on the side where the moment would produce tension, we have as a limit.

$$p_2 = 0 = \frac{1}{A}\left\{ \frac{4HY}{d} - P \right\} \quad \text{or} \quad 4\frac{HY}{d} = P \quad \therefore$$

$$d = \frac{4HY}{P} \text{ . . . . } (2)$$

p573
§381
Cont.

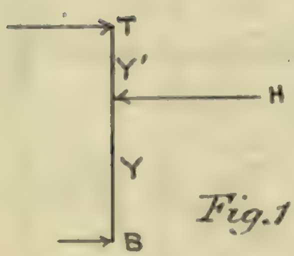


Fig. 1

$$B : T : H :: Y' : Y : Y + Y$$

$$\therefore B = \frac{H'Y}{Y+Y'} \text{ ; } T = \frac{HY}{Y+Y'}$$

$$M_T = \frac{HYY'}{Y+Y'} = M_B \text{ - - - - - - - - - - - - - - - } (3)$$

Using this value for M, instead of HY which was used in getting Eq.(1), we have

$$p_1 = p'' + p' = \frac{1}{A}\left\{ \frac{4HYY'}{(Y+Y').d} \pm P \right\} \text{ . . . . } (4)$$

Suspension Bridge

p574
§382

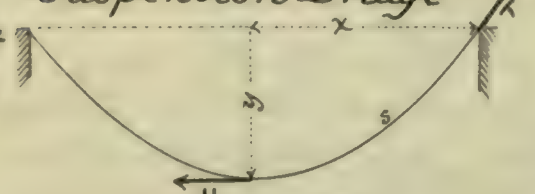


Fig. 2

$\left\{ \text{p190 (13)} \quad s = x + \frac{2y^2}{3x} \text{ nearly} \right\}$

$\left\{ T = H \sec i = H\sqrt{1 + \frac{dy^2}{dx^2}} \right\}$

Let A = sectional area of chain for H

" A' = " " " " " T

p574 $A' : A :: T : H \quad \therefore \; A' = \frac{AT}{H} = A\sqrt{1 + \frac{dy^2}{dx^2}}$

§382 Cont. $C' : C :: wA'\left(x + \frac{2y^2}{3x}\right) : wAx \quad \therefore \; C' = \frac{CA'}{Ax}\left(x + \frac{2y^2}{3x}\right) = \frac{CA}{Ax}\cdot\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\left(x + \frac{2y^2}{3x}\right) \therefore$

$$C' = \frac{C}{x}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\left(x + \frac{2y^2}{3x}\right)$$

$\{x^2 = 2py\} \therefore \; p = \frac{x^2}{2y}$, and $\frac{dy}{dx} = \frac{x}{p} \quad \therefore \left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = \left(1+\frac{x^2}{p^2}\right)^{\frac{1}{2}}$

developing this by the Binomal Theorem, and substituting for p its value, we have, using only two terms.

$$\left(1+\frac{x^2}{p^2}\right)^{\frac{1}{2}} = 1 + \frac{2y^2}{x^2} \text{ nearly}$$

$$\therefore \; C' = \frac{C}{x}\cdot x\left(1 + \frac{8}{3}\frac{y^2}{x^2} + \text{etc.}\right) = C\left(1+\frac{8}{3}\frac{y^2}{x^2}\right) \text{ nearly} \quad \text{---------- (1)}$$

Let A'' = variable section of chain for variable pull T
" x_1 = half span; y_1 = rise; s_1 = length half span
s, x & y variables $\quad x^2 = 2py \therefore p = \frac{x^2}{2y} = \frac{x_1^2}{2y_1}$

$$A'' : A :: T : H \quad \therefore \; A'' = \frac{AT}{H} = A\sqrt{1 + \frac{dy^2}{dx^2}}$$

$$C'' : C :: \int_0^{s_1} wA''ds : wAx_1 \quad \therefore \; C'' = \frac{C}{Ax_1}\int_0^{s_1} A\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} ds \; \therefore \; C'' = \frac{C}{x_1}\int_0^{x_1}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}dx$$

$$C'' = \frac{C}{x_1}\int_0^{x_1}\left(1+\frac{dy^2}{dx^2}\right)dx = \frac{C}{x_1}\int_0\left(1+\frac{x^2}{p^2}\right)dx = \frac{C}{x_1}\left(x_1 + \frac{1}{p^2}\cdot\frac{x_1^3}{3}\right) = \frac{C}{x_1}\left(x_1 + \frac{4y_1^2}{3x_1}\right) \therefore$$

$$C'' = C\left(1 + \frac{4y_1^2}{3x_1^2}\right) \quad \text{---------------- (2)}$$

If we assume that a wire cable can safely bear a strain of 15000 lbs. to the square inch, and wish to find the value of C we have; let A = area in sq. inches; f = 15000

$$C = 12\,Ax \times (\text{wt. cu. in. iron}) = 12\,Ax \times 0.277$$

also $Af = H$ multiplying the two equations together

$$CAf = 12\,HAx \times 0.277 \quad \therefore \; C = \frac{Hx \times 12 \times 0.277}{f} = \frac{Hx \times 12 \times 0.277}{15000}$$

$\therefore \; C = \frac{Hx}{4503}$ To find the value assumed in Eq.(3)

we have $4500 \times 12 \times 0.277 = 14958 = f = \frac{H}{A}$

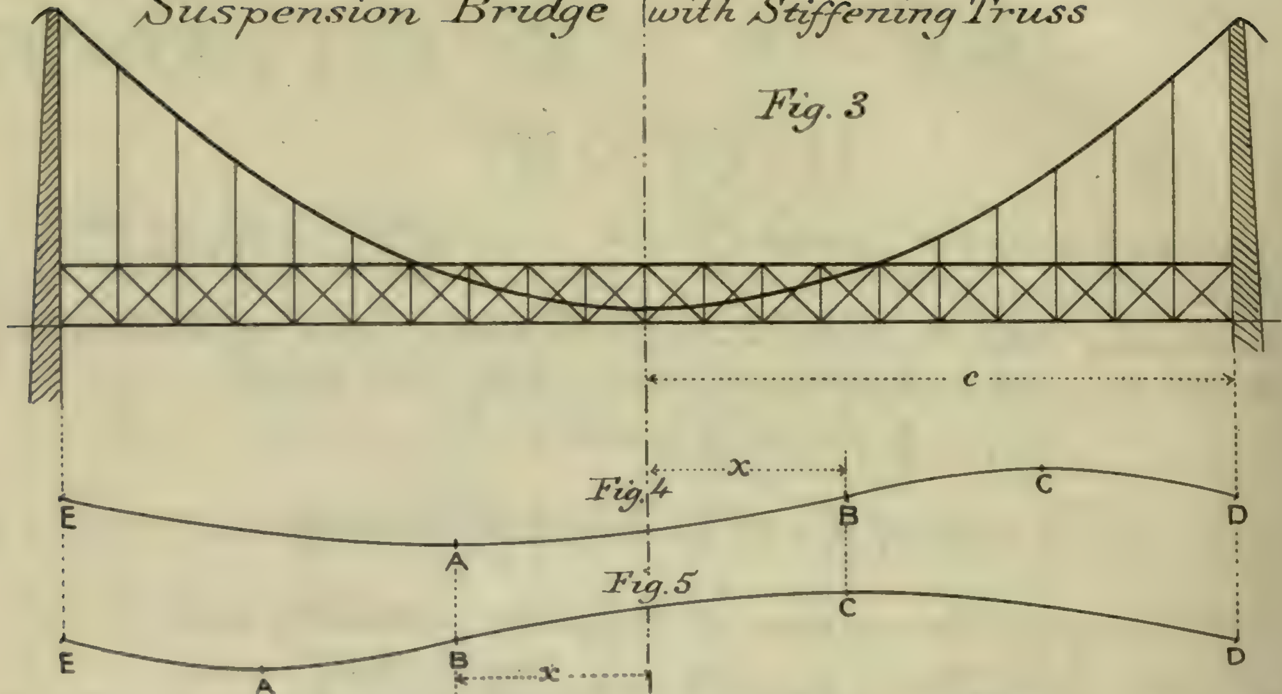
For cable-iron links $C = \frac{Hx}{3000}$ .... Eq.(4); assumes that

$f = 3000 \times 12 \times 0.277 = 9972 = \frac{H}{A} = 10000$ nearly.

p.579 §382 Cont.

VI

Suspension Bridge with Stiffening Truss



In his Applied Mechanics §340, Rankine publishes what he believes to be the first investigation of the theory of this bridge that has appeared in print; and the following is largely a reproduction of that article.

"The weight of the chain itself, being always distributed in the same manner, resists alteration of the figure of the bridge. By leaving it out of account, therefore, an error will be made on the safe side as to the stiffness of the bridge, and the calculation will be simplified".

The Girder partially loaded. Let EB in either of the figs. 4 or 5 represent the length of the loaded part of the stiffening girder, and BD that of the unloaded part; let w be the uniform intensity of the load and x the distance of the point where the load terminates from the centre of the bridge; x being considered as a positive quantity when the loaded part is longer, as in fig. 4, and a negative quantity when the loaded part is shorter, as in fig. 5.

The ends E and D of the beam being fastened, so as to be incapable of vertical displacement, the loaded segment EB is convex downwards, and the unloaded segment BD convex upwards; the loaded segment is in the condition of a beam supported at E and B, and uniformly loaded with the excess of the weight sustained above the force exerted between

p579 the girder and the chain; and the unloaded segment is in
§382 the condition of a beam held down at B and D and loaded with
Cont. an uniformly distributed force acting upward, being that exerted between the girder and the chain. The greatest moment of flexure of each segment is at its middle point, being A for the loaded part and C for the unloaded part.

The length of the loaded segment being

$$EB = c + x$$

its gross load is $W = w(c+x)$

and the intensity of the force exerted between the girder and the chain,

$$w' = \frac{w(c+x)}{2c} \quad \text{(a)}$$

This is the intensity of the upward load on the segment BD whose length is $c-x$; and consequently, according to table p. 246 §161, the greatest moment of flexure of that segment, at C, is

$$M_c = \frac{w'(c-x)^2}{8} = \frac{w(c+x)(c-x)^2}{16\,c} \quad \text{(b)}$$

The amount of upward force exerted between the chain and BD is

$$W' = w'(c-x) = \frac{w(c^2-x^2)}{2c}; \quad \text{(c)}$$

and this also is the amount of the net load on EB, being the excess of the gross load above the part borne by the chain. The half of this quantity

$$F = \frac{W'}{2} = \frac{w(c^2-x^2)}{4c} \quad \text{(d)}$$

is the value of the supporting force exerted by the pier against girder at E, of the shearing force between the two divisions of the girder at B, and of the downward force by which the end D of the girder is held at its point of attachment to the pier.

The intensity of the net load on EB is

$$w + w' = \frac{w(c-x)}{2c}; \quad \text{(e)}$$

and the length of that segment being being $c+x$, its greatest moment of flexure, according to, table, p. 246 §161, is

$$M_a = \frac{(w-w')(c+x)^2}{8} = \frac{w(c+x)^2\cdot(c-x)}{16\,c} \quad \text{(f)}$$

By the usual mode of finding maxima and minima, it is easily ascertained, that the greatest moment of flexure of the loaded division of the girder occurs when $x = \frac{c}{3}$; or when $\frac{2}{3}$ of the beam is loaded; and the greatest moment of flexure

p579 of the unloaded division of the girder, occurs when $x = -\frac{c}{3}$, or
§382 when $\frac{2}{3}$ of the beam are unloaded, and further that those
Cont. two greatest moments are of equal magnitude and opposite in direction, viz.

$$\text{max. } M_A = -\text{max } M_c = \frac{2wc^2}{27}; \text{ ---------- ----(g)}$$

To find max. M_A from Eq. (f)

$$M_A = \frac{w(c+x)^2(c-x)}{16c}$$

$$\therefore \frac{dM_A}{dx} = \frac{w}{16c}\left[2(c+x)(c-x) - (c+x)^2\right] = 0 \quad \therefore 2(c-x) - (c+x) = 0 \quad \therefore x = \tfrac{1}{3}c$$

from Eq. (b) $M_c = \frac{w(c+x)(c-x)^2}{16c}$ $\therefore \frac{dM_c}{dx} = \frac{w}{16c}\left[(c-x)^2 - (c+x)(c-x).2\right]$

$$\therefore c - x - (c+x).2 = 0 \quad \therefore -c - 3x = 0 \quad \therefore x = -\tfrac{1}{3}c$$

In Eq. (f) put for x its value (when M_A is a max.) $\frac{1}{3}c$

$$\text{max. } M_A = \frac{w(c+\frac{1}{3}c)^2(c-\frac{1}{3}c)}{16c} = \frac{w(\frac{4}{3}c)^2 \times \frac{2}{3}c}{16c} = \frac{2wc^2}{27}$$

$$\text{max } M_c = \frac{w(c-\frac{1}{3}c)(c+\frac{1}{3}c)^2}{16c} = \frac{w\frac{2}{3}c \times (\frac{4}{3}c)^2}{16c} = \frac{2wc}{27}$$

From Eq. (d) we see that the max. shearing force occurs when $x = 0$, or when the rolling load covers half the bridge.

$$\therefore \text{max. } F = \frac{wc}{4} \text{ ------------------- (h)}$$

If the girder is hinged at the centre we have two independent half trusses, and the greatest strain on either will be when the load covers half the bridge. When thus loaded the continuous truss and the hinged truss would act in exactly the same way. If then in Eqs. (f) and (b) we make $x = 0$ we get

$$M_A = M_c = \frac{wc^2}{16} \text{ ---------------- (i)}$$

If, to conform with the notation in the text, we write x for c, w' for w, M for M_A or M_c in equations (i) and (h) we have

$$M = \frac{w'x^2}{16} \text{ ---------------- (7)}$$

$$F = \frac{w'x}{4} \text{ ---------------------- (8)}$$

If the girder is continuous we use equation (g)

$$M = \frac{2w'x^2}{27} = \frac{w'x^2}{14} \text{ (nearly) ---------(9)}$$

p583 §383

$$W = s_1 W' + s_2 B \quad \therefore \quad W' = \frac{W - s_2 B}{s_1} \quad \text{--------} \quad (1)$$

$$W : s_2 B :: L : l \quad \therefore \quad L = l \frac{W}{s_2 B} \quad \therefore$$

$$L = l \frac{s_1 W' + s_2 B}{s_2 B} = l \left(1 + \frac{s_1 W'}{s_2 B}\right) \quad \text{--------} \quad (2)$$

$$\frac{s_1 W'}{s_2 B} = \frac{L - l}{l} \quad \therefore \quad \frac{s_2 B}{s_1 W} = \frac{l}{L - l} \quad \therefore \quad \frac{B}{W'} = \frac{s_1}{s_2} \cdot \frac{l}{L - l} \quad \text{--------} \quad (3)$$

$\{s_1 = 6 \; ; \; s_2 = 3\}$ then $L = l\left\{1 + \frac{6}{3}\frac{W'}{B}\right\} = l\left\{1 + 2\frac{W'}{B}\right\}$ -------- (4)

$$\frac{B}{W'} = \frac{6}{3} \cdot \frac{l}{L - l} = \frac{2l}{L - l} \quad \text{--------} \quad (5)$$

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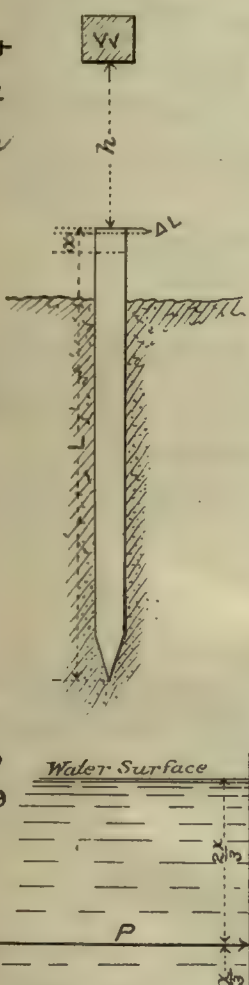
p604 §402 Note



Let F = Force between hammer and pile

$\left\{\frac{\Delta L}{L} = \alpha = \frac{F}{\mathcal{E}S}\right\}$ The force F is nothing when the weight first touches the pile, and increases to = P, when the pile moves.

The top of the pile is compressed a distance ΔL, while at the bottom there is no compression; so that the mean compression is $\frac{\Delta L}{2}$, or, what amounts to the same thing, the effective length is $\frac{L}{2}$ $\therefore$ The work done in compression $= \frac{P}{2} \cdot \frac{\Delta L}{2} = \frac{P.LP}{\cancel{4}\,\mathcal{E}S}$ 2

The work done in driving is Px

$$\therefore Wh = \frac{P^2 L}{2\mathcal{E}S} + Px$$

Solving this quadratic for P we get its value.

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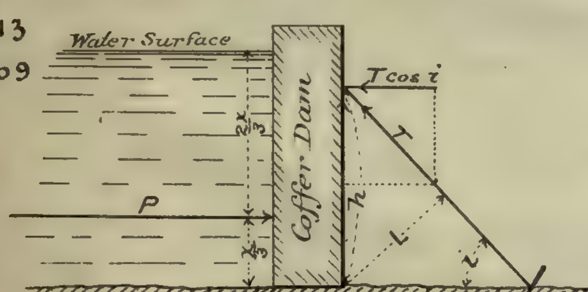
p.613 §409



$$P = wb \cdot \tfrac{1}{2}x \cdot x = \frac{wbx^2}{2} \quad \text{--------} \quad (1)$$

$$M = \frac{wbx^2}{2} \cdot \frac{x}{3} = \frac{wbx^3}{6} \quad \text{--------} \quad (2)$$

$$M = T \cos i \times h \quad \therefore \quad T = \frac{M}{h} \sec i \quad \text{--------} \quad (3)$$

---

p644 §431A

Let φ = retardation due to resistance of brakes.

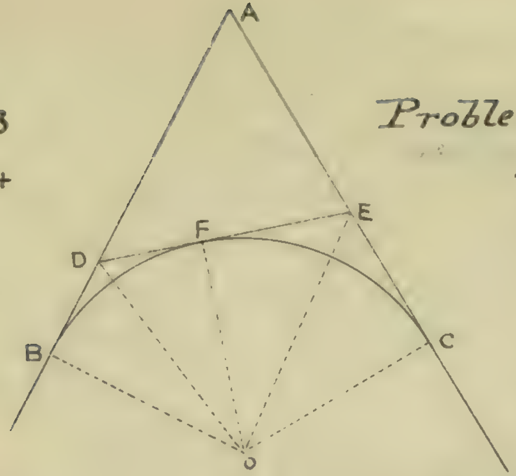
" R = resistance produced by brakes.

Then $\frac{R}{W} = f'$ and

$$W : R :: g : \varphi \quad \therefore \quad \varphi = \frac{R}{W} \cdot g = f'g$$

$$v^2 = 2\varphi h = 2f'g h = 64.4 f' h \quad \therefore \quad h = \frac{v^2}{64.4 f'} \quad \text{-----} \quad (1)$$

p653
§434



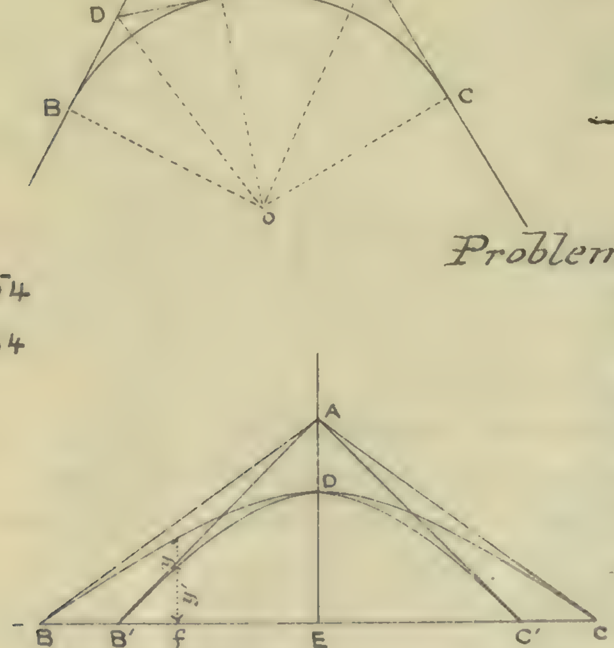
Problem I $\angle BOF = \angle ADF \therefore$

$\angle DOF = \angle \frac{1}{2} D \therefore DF = FO \tan \frac{1}{2} D$

So $FE = FO \tan \frac{1}{2} E \therefore$

$$FO = R = \frac{DE}{\tan \frac{1}{2} D + \tan \frac{E}{2}} \qquad (9)$$

p.654
§434



Problem II The simple sine curve has for its equation

$y = \sin x$ and is represented by $B'DC'$ where the angle $B'AC'$ = a right angle (since $\frac{dy}{dx} = \cos x = 1$, when $x = 0$, which gives $AB'E = B'AE = 45^\circ$)

Now in this curve the radius of curvature =

$$r = \frac{(1+p'^2)^{3/2}}{p''} = \frac{(1+\cos^2 x)^{3/2}}{-\sin x}$$

At D when $x = BE = 90^\circ$, this $r = 1 = DE$, again $B'E = AE = 90^\circ = 1.5707$
$DE = 1 \therefore AD = .5707 \therefore \frac{AE}{AD} = 2.75193$

or $AE = 2.75193 \times AD$ and $AB' = 2.75193 \times AD \sec \frac{1}{2} B'AC'$

$$\text{and } \frac{DE}{AD} = \frac{1}{.5707} = 1.75193 \qquad (11)$$

Now when the tangents make an angle BAC a "sine curve" may be drawn between them by the equation

$$y = DE \sin \frac{90^\circ x}{BE} \qquad 12$$

In this case DE no longer = the radius of the circle but is still the maximum ordinate; AE remains unchanged, and $BE = 2.75193 \times AD \times \tan \frac{1}{2} BAC$. In this case

$$\frac{dy}{dx} = 90^\circ . \frac{DE}{BE} \cos \frac{90^\circ x}{BE} \text{ and } \frac{d^2y}{dx^2} = \left(\frac{90^\circ}{BE}\right)^2 \times DE \sin \frac{90^\circ x}{BE}.$$

$$\therefore r = \frac{\left[1+\left(\frac{90^\circ}{BE}\right)^2 . \overline{DE}^2 \cos^2\left(\frac{90^\circ x}{BE}\right)\right]^{3/2}}{\left(\frac{90^\circ}{BE}\right)^2 . DE \sin\left(\frac{90^\circ x}{BE}\right)} = (\text{when } x = BE \text{ at } D) \frac{1}{\left(\frac{90}{BE}\right)^2 . DE}$$

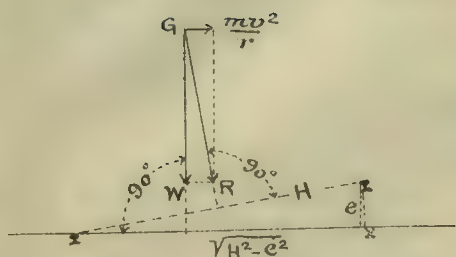
$$\therefore r = \frac{\overline{BE}^2}{DE(1.5707)^2}$$

But $BE = 2.75193\ AD \tan \frac{1}{2} BAC$; and $AD = \frac{DE}{1.75193} \therefore BE = DE(1.5707) \tan BAC$

$$\therefore \overline{BE}^2 = \overline{DE}^2 (1.5707)^2 \tan^2 \frac{1}{2} BAC. \qquad (13)$$

$$\text{Hence } r = DE \tan^2 \frac{1}{2} BAC \qquad (14)$$

p.649 §434



H = gauge.
e = Elevation of outer rail, in feet.
$\frac{mv^2}{r}$ = centrifugal force.
m = mass = $\frac{W}{g}$

From similar triangles

$$e : \sqrt{H^2-e^2} :: \frac{mv^2}{r} : W$$

Since e^2 is small compared with H^2, it may be dropped from under the radical, and we have

$$e = \frac{mv^2H}{rW} = \frac{v^2H}{rg} = \frac{v^2H}{32.2\,r} \text{ - - - - - - - - - - } (3)$$

1 mile per hour = 5280 ft in 3600 seconds = $\frac{5280}{3600}$ ft. in 1 second

Therefore if velocity is expressed in miles per hour we must multiply by $\frac{5280}{3600}$ to reduce to feet per second;

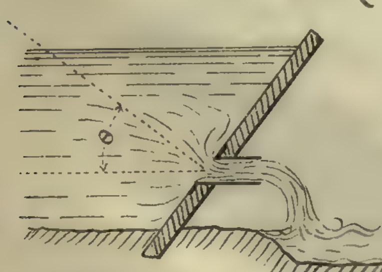
$$\therefore\ e = \frac{\left(\frac{5280}{3600}\right)^2 V^2 H}{32.2\ r} = \frac{V^2H}{15\ r} \text{ nearly - - - - - - - - - - - - } (4)$$

p.677 §447 The coefficient of velocity through a thin plate = .97 substitute this in equation (3), § 446, and calling the theoretical velocity v', we have

$$0.97\,v' = 8.025\sqrt{\frac{v'^2}{64.4(1+F)}} \quad \therefore\ F = 0.054 \text{ - - - - - - - - - } (1)$$

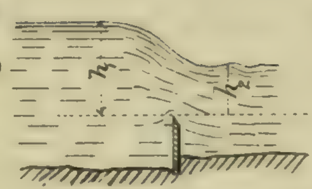
For a straight cylindrical mouthpiece the coefficient of velocity = .813 which is the same as the coef. of discharge, since there is no contraction, Substitute in Eq.(3), § 446, and we get

$$(.813)^2 = \frac{1}{1+F} \quad \therefore\ F = 0.505 \text{ - - - - - - - - - - - - - - } (2)$$



The coëf. of mouthpieces is the same as for short tubes. Equation (3) is an empirical formula and is the result of extended experiments by Julius Weisbach (see Theoretical Mechanics). Weisbach says the coëf is greater, than for those at right angles, because the direction of the water is changed.

$$F = 0.505 + 0.303 \sin\theta + 0.226 \sin^2\theta \text{ - - - - - - - - } (3)$$

p684 §449 

Here the flow due to that portion of the notch which is not drowned is

$$Q' = 8.025\ c\int_a^{h_1-h_2} b\sqrt{h}\,.dh$$
$$= 5.35\,c\,b\,(h_1-h_2)^{3/2}$$

Of the drowned part, it is

$$Q'' = 8.025\ c\,A\sqrt{h} = 8.025\ c\,h_2\,b\sqrt{h_1-h_2}$$
$$= 5.35\ c\,b\,h_2 \cdot \tfrac{3}{2}\sqrt{h_1-h_2}$$

Adding together we get

$$Q = 5.35\ c\,b\left(h_1+\frac{h_2}{2}\right)\sqrt{h_1-h_2} \text{ - - - - - - - - - } (7)$$

p686 §450 From Eq.(1)

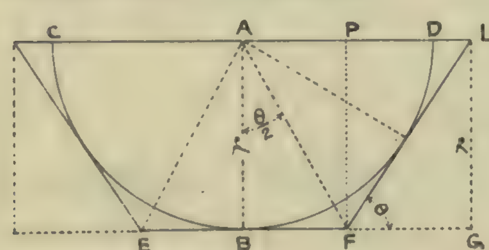
$$v = \frac{Q}{0.7854\,d^2} = 8.025\sqrt{\frac{hd}{4fl}} \quad \text{and } d = \left(\frac{4flQ^2}{39.73\,h}\right)^{1/5}$$

Let $f'' = f' + \Delta f'$ then $d = d'\left(\frac{f'+\Delta f'}{f'}\right)^{1/5} = d'\left(1+\frac{\Delta f'}{f'}\right)^{1/5} = d'\left(1+\frac{1}{5}\frac{\Delta f'}{f'} + \text{etc}\right)$

$= d'\left(1+\frac{1}{5}\frac{\Delta f'}{f'}\right)$ nearly $\therefore\ d = d'\left(1+\frac{1}{5}\frac{f''-f'}{f'}\right) = d'\left(\frac{4}{5}+\frac{f''}{5f'}\right)$ - - - - - - - - (7)

$$d = d'\left(\frac{h+h''}{h}\right)^{1/5} = d'\left(1+\frac{h''}{h}\right)^{1/5} = d'\left(1+\frac{1}{5}\frac{h''}{h}\right) \text{ nearly - - - - - - - - - - - - - } (8)$$

Neville's Section

p688 §451 

$FL = r\,\text{cosec}\,\theta$

Area $AFL = \frac{1}{2}r^2\,\text{cosec}\,\theta$

" $ABF = \frac{1}{2}r^2\tan\frac{1}{2}\theta$

Total area $= r^2\left(\text{cosec}\,\theta + \tan\frac{\theta}{2}\right)$

Wetted perimeter $= 2\,FL + 2\,BF$

$= 2r\,\text{cosec}\,\theta + 2r\tan\frac{\theta}{2}$

$$\therefore\ m = \frac{r^2\left(\text{cosec}\,\theta+\tan\frac{\theta}{2}\right)}{2r\left(\text{cosec}\,\theta+\tan\frac{\theta}{2}\right)} = \frac{1}{2}r$$

$$n = \frac{A}{m^2} = \frac{r^2\left(\text{cosec}\,\theta+\tan\frac{\theta}{2}\right)}{\frac{1}{4}r^2} = 4\left(\text{cosec}\,\theta+\tan\frac{\theta}{2}\right)$$

From Equations (2) and (3) we have

$$m' = \frac{Q'^2}{8512\,A^2 i} = \left[\text{since } A^2 = n^2 m'^4\right], \frac{Q'^2}{n^2 m'^4\,8512\,i}$$

$$\therefore\ m' = \left(\frac{Q'^2}{8512\ n^2 i}\right)^{\frac{1}{5}} \text{ - - - - - - - - - - - - - - - } (5)$$

For Equation (7) see Equation (7) of § 450

Errata

Rankine's Civil Engineering Text 19th Edition 1894.

p 47 Eq.(3) read $m'' = \frac{mm'}{m\cos^2\theta + m'\sin^2\theta}$
for $m'' = m\cos^2\theta + m'\sin^2\theta$

148 Eqs.(7) read $M_1 = 0$, $M_2 = 0$
for $M = 0$, $M = 0$

156 23rd line read OD for OA

159 XVI read $\frac{z_1}{r}$ for $\frac{z'}{r}$

160 XIX read $W = \frac{2\pi w}{3}(r^3 - r'^3)(\cos\alpha - \cos\beta)$

170 14th line read the Arrows BB
for the Angles BB

205 Eq.(5) read $\sqrt{\frac{p_y'}{p_x'}}$ for $\sqrt{\frac{p_y}{p_x'}}$

207 Eq.(3) read $\frac{p}{c}$ for $\frac{p'}{c}$ and $p_y' = \frac{H'}{r}$ for $p_y = \frac{H'}{r}$

208 Eq.(6) last term read in the denominator
$(B^2 - A^2)$ for $(A^2 - B^2)$

210 Eq.(7) read $H = w\int_{x_0}^{x_1} x\,dx$ for $H = w\int_{x_0}^{x_1} p\,dx$
" (9) " $w\int_{x}^{x_1} w\,dx$ for $w\int_{x}^{x_1} p\,dx$

213 Eq.(1) " $c = \sqrt{\frac{p_y}{p_x \cos j}}$. for $c = \sqrt{\frac{p_y}{p_x}}$

220 5th line read fig. 123, for fig. 95

234 VI read $h^2 - h'^2$ for $h'^2 - h'^2$
7th line from bottom read fig. 125 for 121
about middle read $P \geqq fS \div 2$ for $P < fS \div 2$

247 Table value of m for N even
read $\frac{N}{8(N-1)}$ for $\frac{1}{8}$

280 10th line read $\frac{2n''}{6}$ for $\frac{n''}{6}$

284 8th " " less for greater
Eq.(5) " f' for f

285 In value v_1 Eqs(7),(8) read f' for f

290 1st line middle Colum of Table
read x'^2 for x^2

293 17th line from bottom read x for x'

296 Case XVII of table should be omitted as it refers to an article which appeared in earlier editions but is omitted, at least, in all after the 8th edition

Text continued

308 Eqs.(37 B),(38B) read 0.4 for 0.192

319 Eq.(1A) read $\geqq$ for $\leqq$.

378 1st line " p.360 for p.36

406, 407 read r for q'

423 Eq.(3) last term read $-\frac{w'c}{2wA}$ for $-\frac{w'}{2wA}$

550 16th line read pp.240 for pp.230

These Notes on Rankine's C.E.

p.2 Eq.(7)&(7A) in denominator read
Ab for Bc

3 4th line from bottom read Cx^2 for Cy
11th line read $y_2 = b_2$ &c for $y_3 = b_3$ &c.

4 20th & 21st lines read C_1H for CH

6 2nd line from bottom read m_1 for m'

7 10th " " " " $\begin{cases} O' = \\ P' = \end{cases}$ for $\begin{cases} O = \\ P = \end{cases}$

9 14th " read $\cos\frac{C}{2}$ for $\frac{C}{2}$

10 10th " " DNB " ANB
12th " " DMB " AMB
5th " from bottom DPZ for APZ

23 2nd line from " read $\sqrt{2p} = \frac{y_1}{\sqrt{x_1}}$

24 2nd " " " " $\frac{2\sin^2\frac{\theta}{2}}{\theta}$ in numerator

25 X in the value of x_0 read in the denominator
r'^2 for r'^3

26 6th line read z_0 for z
3rd " from bottom read z_1 for z_1' in value of z_0

27 XVI in value z_0 read z_1 for z'

28 4th line read ρ for r

29 5th " from bottom read $\sin\theta$ for $\sin\alpha$

30 Eq.(10) in denominator read x_1^2 for x^2

44 5th & 6th lines the last term in the brackets should be divided by 2

46 Eq. (b) read $1 + \frac{s^2}{m^2}$ for $1 - \frac{s^2}{m^2}$

47 " (9) " XT " BT

49 13th line read (6) for (5)
also $\cos(i' \pm j)$ for $\cos(i \pm j)$

Errata

page

52 Eq. (5) read $\frac{O'B'}{O'A'}$ for $\frac{OB'}{OA'}$
" (6) " $p' = p\frac{r'}{r''}$ for $p = p\frac{r'}{r''}$

53 " (6) read p_x for p_y Eqs. (7) & (8) read p_y for p_x

58 8^{th} line from bottom read $\operatorname{cosec}^2 i$ for $\sec i$

63 Middle, read $\left\{2\int_r^R q\, r'dr = p_1 r - p_2 R\right\}$

66 VII under $\frac{x_1 S}{I}$ read $\frac{6h}{h^3 - h'^3}$
VIII " J " $\frac{\pi}{64}(h^4 - h'^4)$

67 X read $I = \iint x^2\, dx\, dy = \iint (x_0^2 + 2x_0 x' + x'^2)\, dx\, dy$
$= x_0^2 \iint dx'\, dy' + 2x_0 \iint x'\, dx'\, dy' + \iint x'^2\, dx'\, dy'$
11^{th} line read OX' & OY' for OX & OY

71 3^{rd} line below the figures read
$P_1'' : w'(l-x) :: \frac{l-x}{2} : l \therefore P_1'' = \frac{w'(l-x)^2}{2l}$
and the last member of each equation down to and including Eq. 3 should be divided by 2

72 last member of equation at bottom of page, read $\frac{(c^2 - x^2)y}{2x(y_1^2 - y^2)}$

73 Eq. (3) read $-\frac{dy}{dx} = \frac{c^2 - x^2}{2x(y_1^2 - y^2)} \pm \sqrt{1 + \frac{(c^2 - x^2)^2 y^2}{4x^2(y_1^2 - y^2)^2}}$

74 Eq. (2) 1^{st} read $y_1 = \frac{h'}{2}\left(1 + \frac{A_1}{A}\right)$
also last term in Eq. (b) read $\frac{1}{4} A_2 h_1^2$ for $\frac{1}{4} h_1^2$

77 6^{th} line read $\left\{y_a = \frac{h'}{2}\left[\frac{A + A_1}{A}\right]\right\}$ for $\left\{y_a = \frac{h'}{2}\left[\frac{A + 2A_1}{A}\right]\right\}$

78 Case II read Uniformly Loaded for Loaded at B

79 Case IV last equation read $\frac{x^2}{c^2} + \frac{y^2}{h^2} = 1$ for $\frac{x^2}{h^2} + \frac{y^2}{c^2} = 1$

93 Eq. (1) read $\iint \frac{dx^2}{EI} = n$ for $\iint \frac{d^2x^2}{EI} = n$